Construction Methodologies for Implied Volatility Surfaces

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Abstract

This research paper will provide a kit of "everything that needs to be known" about volatility to anyone coming from any scholastic background. It will allow the reader to understand how to use the Heston and Stochastic Alpha Beta Rho (SABR) models through MATLAB and the Monte Carlo and Black-Scholes processes through Excel. Another objective is to bring into practical terms all the theoretical formulas and concepts and see which of the previously stated construction methodologies performs best under various circumstances.

There will firstly be an introduction to the concept of volatility. The importance of the implied volatility surface will then be shown and there will be explanations of the various construction methodologies for implied volatility surfaces, varying form local stochastic volatility models to Levy processes and parametric representations.

In the empirical experimentations section, the Heston, SABR and Monte Carlo models will be put at comparison and will be examined in terms of correctly portraying the market implied volatility surface and market option prices for five equities. There will additionally be an attempt to capture the dynamics of the implied volatility surface through the examination of how the market and models' surfaces evolve throughout time.

We will see that the SABR model is the most appropriate construction methodology with respect to both estimating implied volatility surfaces and calibrating option prices among the three analysed throughout the empirical section. However, it must also be noted that better implied volatility estimation does not necessarily mean better option price calibration, and vice-versa. Lastly, a model that is more precise than the others does not guarantee that it is precise when compared to market data.

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1 Introduction: what is volatility?

1.1 Definition of volatility

Volatility takes various forms. It defines "how risky an investment is", or also "the likelihood of a stock crashing". We might also talk about stochastic volatility, which is volatility that can take different values in option pricing models (and that hence is randomly distributed). Realised volatility (or historical volatility) measures how the market has actually changed in the past. It "a statistical measure of the dispersion of returns for a given security or market index"[5].

It could as well be implied volatility, which is the volatility of an option that should persist shall various assumptions about the Δ (the sensitivity of an option's price to changes in the underlying's price), Γ (the first derivative of Δ, or also the second derivative of the option's price with respect to the spot price), ρ (the sensitivity of an option's price relative to interest rates), θ (the sensitivity of an option's price relative to time to expiration of the option), strike price K and price of the derivative P are satisfied in the Black-Scholes model. It is an expectation of "how volatile the market can be in the future"[6]. It is "the σ input into the BSM formula that generates the market observed option price"[7]. Let us quickly remember the assumptions of the Black-Scholes model:

1. The asset's asset price follows a Geometric Brownian Motion with σ and r_f constant.
2. Short selling of securities is allowed.
3. No frictions.
4. No dividends.
5. No arbitrage opportunities.
6. Continuous markets.
7. r is constant for every maturity.[8].

When talking about implied volatility, it can be considered as an alternative way of pricing a derivative. Traders often execute transactions based on their beliefs of implied volatility. When implied volatility is relatively high, there are expectations of a big market movement. When it is low, the market (or better, the price of the specific derivative) is expected to remain relatively stable.

Commonly marked with a σ, which stands for standard deviation, volatility is a key aspect of pricing options and, as we will see throughout this research paper, multiple models ave tried to capture the exactitude of implied volatility of options with respect to the θ and the K of the option. Traders often base their trades on implied volatility instead of on the option price itself.

There is a positive relationship between the volatility of an option and its price. Indeed, the higher the volatility of an option, the higher the probability of that option finishing in-the-money.

1.2 Why is volatility so important?

As stated previously, there are different types of volatility. Throughout this research paper we will be focusing a lot on implied volatility, which is calculated form the Black-Scholes-Merton model. But why use implied volatility instead of using option prices directly when quoting? It is more straightforward to express a view with respect to implied volatility than with respect to option prices. Indeed, implied volatility does not vary when there is a variation in the spot price and/or in the strike and/or in the option premium. Therefore, implied volatility is not dependent upon intrinsic value; while option prices are. However, the intrinsic value has no real information value. Implied volatility, additionally, "has the normal return distribution (Black-Scholes-Merton model) as a benchmark"[9].

As an example, a call with 2 years to expiration, strike price \$50 priced at \$2.5 cannot really be compared to another call with 2 months to expiration, strike at \$45 and priced at \$3. However, if the first one has an implied volatility of 30% and the second one an implied volatility of 50%, the comparison is clearer.

Moreover, if options are quoted with respect to a positive implied volatility surface, the type 1 no-arbitrage conditions will be directly insured[10], where the type 1 no-arbitrage condition is the no-arbitrage condition between European options of a predetermined strike and a predetermined maturity vs. the underlying cash[11].

Lastly, there is also the technological aspect to it: when there is no options order flow, there is no necessity to update the implied volatility surface as often as option prices[12].

1.3 What affects volatility?

When there is a disequilibrium between demand and supply for an underlying, the implied volatility of the derivative changes too.

When there is very high demand, the price of the security rises, as well as its implied volatility. The opposite is true when there is excess supply, and the option becomes cheaper.

[5] http://www.investopedia.com/terms/v/volatility.asp

[6] https://www.tradeking.com/education/options/what-is-implied-volatility

[7] http://faculty.baruch.cuny.edu/lwu/9797/Lec8.pdf

[8] http://www2.math.su.se/matstat/reports/seriec/2011/rep7/report.pdf

[9] Carr and Wu - *A new simple approach for constructing implied volatility surfaces. (2011)*

[10] Ibidem

[11] Ibidem

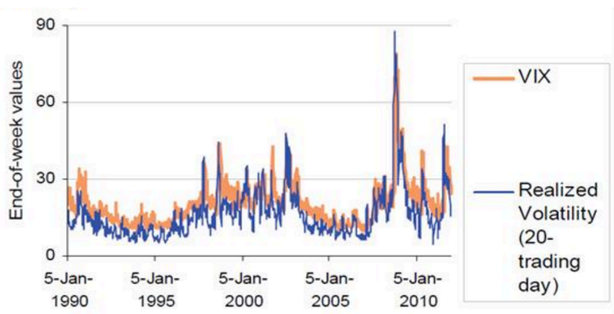
[12] Ibidem

The time value of a derivative is also a key factor determining the total implied volatility of the derivative itself. Indeed, taking the example of an option, the longer is the time to expiration, the higher the chance that the price of the option has in order to vary throughout time, which means higher implied volatility[13].

Lastly, the strike price K of an option is a vital aspect of determining how much volatility an option has. The closer the option's price is to the strike price, the higher the volatility. Hence, at-the-money options have the highest implied volatilities.

1.4 Realised volatility vs. Implied volatility: empirical evidence

Figure 1: S&P's realised vs. implied volatility, 1990 - 2010 [14]



As you can see, when the market is not in turmoil, realised volatility is actually lower than implied volatility. The opposite is true when there are market crashes: see as examples the Russian and South East Asian crises of the late 90s and the Financial Crisis of 2008.

1.5 Volatility smiles, skews and the volatility term structure

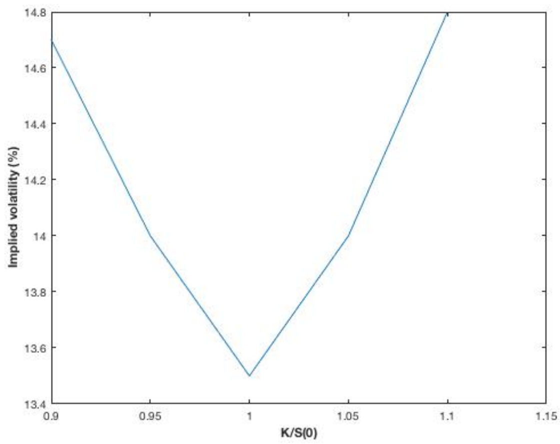
What models do traders use when pricing options? Do they really use the Black-Scholes-Merton model or do they use a slightly different version of it? Are market prices of options very different or close to those predicted by Black-Scholes-Merton? "Are the probability distributions of asset prices really lognormally distributed?"[15] In order to answer these questions, a closer look at the concept of volatility smile has to be given.

The key takeaway is that traders use a modified version of the Black-Scholes-Merton: one where they allow volatility to be stochastic and dependent on K and θ. Let us see how volatility smiles work for both equity and foreign currency options.

As stated previously, implied volatility is a function of both K and θ. When implied volatility is a function of exclusively the strike price, we observe empirically either volatility smiles or volatility skews.

The first one consists of a curve for which deep in-the-money and deep out-of-the-money options have far higher implied volatilities than at-the-money options. An example of the volatility smile is that of FX options.

Figure 2: Volatility smile from Table 1, 1 year to maturity



The volatility skew is instead present in equity options and equity index options markets. It consists of a downward sloping convex curve where implied volatility decreases as the spot price of the options increases. Therefore, it is very low for in-the-money calls and out-of-the-money puts. An example is how in equity derivatives markets "extreme events" (such as a stock crash) are associated with higher implied volatilities, where a bearish market is riskier (and hence has higher volatility) than a bullish one[16].

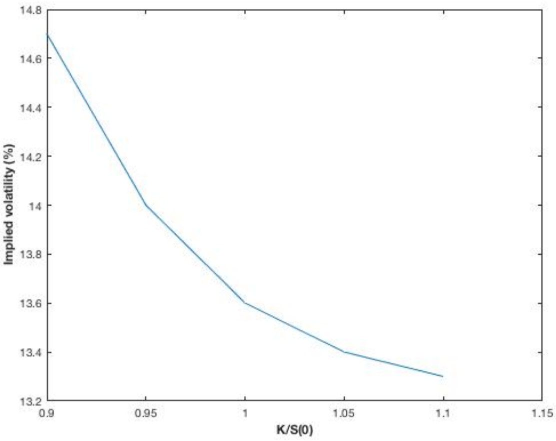
[13] http://www.investopedia.com/terms/i/iv.asp

[14] http://www.spvixviews.com/wp-content/uploads/2012/02/Exhibit-13-SP-500-Implied-Volatility-vs.-Subsequent-Realized-Volatility.jpg

[15] J. C. Hull - *Options, Futures, and other Derivatives Eighth Edition. (2012)*

[16] L. Marroni, I. Perdomo - *Pricing and Hedging Financial Derivatives - A guide for Practitioners. (2014)*

Figure 3: Volatility skew example



Reasons for a volatility skew include: leverage (stocks are more volatile at lower prices than higher ones); strong negative correlation between volatility and spot price moves; jumps in prices are more often seen downwards than upwards; supply and demand, as described previously. The volatility skew has been existing since the 1987 stock market crash[17].

We also need to take into account the T relationship, where T stands for time to maturity: the term structure of volatility plots implied volatility against T. Usually there is a direct relationship between the two variables. In other words, the lower is the time to expiration, the higher is the implied volatility. Therefore, the relationship between implied volatility and maturity can be thought of as a decreasing convex curve.

Thus, traders do not use the exact forms of the Black-Scholes-Merton model, because of various reasons: firstly they allow for the strike price and time to maturity to vary, and secondly they assume a distribution for the asset (be it FX, equities, etc.) prices that is not lognormal.

While remembering that the implied volatility of European calls is the same as that of European puts, let us recall the put-call parity[18]:

$$p + S_0 e^{-qT} = c + Ke^{-rT} \tag{1}$$

where q is the asset dividend yield. The put-call parity condition is true whatever the distribution of the asset price. Since put-call parity holds for the Black-Scholes model:

$$p_{BS} + S_0 e^{-qT} = c_{BS} + Ke^{-rT} \tag{2}$$

In a no-arbitrage situation then we also have, for market prices:

$$p_{mkt} + S_0 e^{-qT} = c_{mkt} + Ke^{-rT} \tag{3}$$

If we subtract the two equations, we have:

$$p_{BS} - p_{mkt} = c_{BS} - c_{mkt} \tag{4}$$

Thus, if for example the implied volatility of a call option is 15%, then the Black-Scholes call price equals the market price of the call option when a volatility of 15% is used in the BSM model. Hence, given K and T, the correct volatility to use with the Black-Scholes-Merton model to price a European call should be the same as the one that is used to price a European put. Consequently the volatility smile is identical for both European calls and European puts, as well as the volatility term structure[19].

For foreign currency options, empirical evidence is that the lognormal distribution "understates the probability of extreme movements in exchange rates"[20] . The reasons for a smile in foreign currency options are that the volatility of the asset is not constant and that there are occasional jumps. The impact of these two characteristics of foreign currency options depends on T. As the time to maturity increases, the impact of non-constant volatility is more significant while the percentage impact on implied volatility is less so. On the other hand, the impact of jumps on prices and implied volatility becomes smaller and smaller as T augments. Therefore, "the volatility smile becomes less pronounced as option maturity increases"[21].

When analysing equity options, because of what stated previously about leverage, the implied distribution is characterised by a fatter left tail and thinner right tail then the lognormal distribution. Indeed, as the stock price of an asset decreases, the leverage increases, making the stock more risky, with a consequent increase in volatility. Vice-versa, when a firm's equity increases in value, the amount of leverage decreases, making the stock less risky and thus a consequent decrease in volatility. Moreover, another explanation for the volatility skew in equity options is that since the October 1987 crash traders have become more scared of crashes, and they have since then adjusted for such probability when pricing options.

It is important to use K/S_0 instead of K simply when plotting the volatility skew, because if there is a change in the equity price, then the volatility skew tends to move. This way there is much more stability in the skew and hence using such standardised measure allows to compare charts across various maturities and asset classes. Sometimes F_0 is used instead of S_0, since F_0 is the "expected stock price on the option's maturity date in a risk-neutral world"[22]. Other practitioners, such as Wu, use an even more standardised measure

[17] http://faculty.baruch.cuny.edu/jgatheral/impliedvolatilitysurface.pdf

[18] J. C. Hull - *Options, Futures, and other Derivatives Eighth Edition. (2012)*

[19] Ibidem

[20] Ibidem

[21] Ibidem

[22] Ibidem

for the moneyness of the option, which is $\frac{Ln\left(\frac{K}{F_0}\right)}{\sigma\sqrt{T}}$[23], where σ is the at-the-money volatility of the option. Other standardized measures include d_1, d_2 and Δ.

The volatility smile can also be defined as the relationship between the implied volatility and the delta of the option, where usually the at-the-money option is a call option with a delta equal to 0.5 or a put option with a delta equal to -0.5.

The assumption of lognormality of the probability distribution of the underlying asset is made by the Black-Scholes-Merton model, but not by traders. The tails are fatter for FX options and the left tail of an equity option is fatter while its right tail is flatter when compared to the respective lognormal distributions. Fatter left tails are also said to exhibit leptokurtosis. For equity options, since the volatility smile is actually a skew, there is negative skewness in the distribution. Lastly, "for stock indexes the distributions are negatively skewed at both short and long horizons"[24].

Thus, non-flat implied volatility curves show that the distribution of returns of the underlying security is not at all normally distributed[25]. The reason why traders use volatility smiles is because "volatility smiles allow for nonlognormality"[26].

1.6 What is an implied volatility surface?

An implied volatility surface plots the implied volatility of the option of a particular asset as a function of strike price K and time to maturity θ. There are two types of implied volatility surfaces: the one just described is an absolute implied volatility surface. If instead of using K, Δ is used, then the result is a relative implied volatility surface[27].

The volatility surface is a combination of the volatility smile/skew and the term structure of volatility, resulting in a 3-D graph, as Figure 4 shows. As stated previously, it is important to note that implied volatility and time to maturity have a direct relationship when short-dated volatilities are historically low, since there are expectations for the volatilities to increase[28]. The opposite is true when short-dated volatilities are historically high, since there is an expectation for volatilities to decrease.

Figure 4: Implied Volatility Surface developed on MATLAB [29]

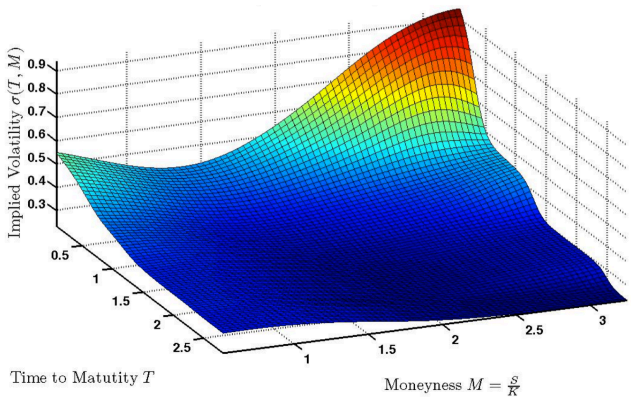


Table 1 shows an example of implied volatility surface data (with the respective graph, Figure 5).

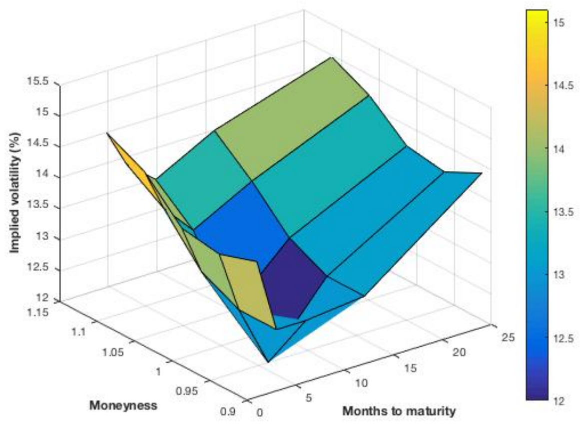
Table 1: Volatility Surface example

	K/S_0				
	0.90	0.95	1.00	1.05	1.10
1 month	14.2	13.0	12.0	13.1	14.5
3 month	14.0	13.0	12.0	13.1	14.2
6 month	14.1	13.3	12.5	13.4	14.3
1 year	14.7	14.0	13.5	14.0	14.8
2 year	15.0	14.4	14.0	14.5	15.1

Usually some of the data correspond to options for which trustworthy market prices can be retrieved. As a consequence, the implied volatilities for these specific prices and maturities are calculated directly, while the rest of the table is completed through interpolation. As an example, a 2 month option with a K/S_0 ratio of 1.10 would be interpreted by the financial engineer as having a value between 14.2 and 14.5; for example, using the average of the two numbers, 14.35%. This is the number that would be used in the Black-Scholes-Merton formula. So, "some points on a volatility surface for a particular asset can be estimated directly because they correspond to actively traded options"[30].

[23] http://faculty.baruch.cuny.edu/lwu/9797/Lec8.pdf
[24] Ibidem
[25] Ibidem
[26] J. C. Hull - *Options, Futures, and other Derivatives Eighth Edition. (2012)*
[27] https://en.wikipedia.org/wiki/Volatility smile
[28] J. C. Hull - *Options, Futures, and other Derivatives Eighth Edition. (2012)*
[29] http://www.mathworks.com/matlabcentral/fileexchange/23316-volatility-surface/content/VolSurface.m
[30] http://www-2.rotman.utoronto.ca/ hull/downloadablepublications/DHSPaperdraft7.pdf

Figure 5: Volatility Surface of Table 1



Certain financial engineers define the volatility smile as the relation between implied volatility and the following $\frac{1}{\sqrt{T}} Ln\left(\frac{K}{F_0}\right)$ instead of that between the implied volatility and K[31].

The choosing of the model is fundamental to the shape of the volatility surface. If traders chose a model different to the Black-Scholes-Merton one, then, even with the shape of the smile changing, "the dollar prices that are quoted in the market would arguably not change appreciably"[32]. What has to be underlined is that "models have the most effect on the pricing of derivatives when the derivatives do not trade actively in the market"[33], such as certain exotic options.

Volatility surfaces are used by traders to value European options when the price of such options cannot be directly observes in the market[34]. There is however a problem with exotic options such as barrier options, because, as Hull and Suo underline, the approach cannot be really extended to path-dependent exotic options. Hence, some model risk exists.

Another use of the volatility surface is to hedge against changes in asset prices.

1.7 What does the implied volatility surface communicate to us?

Let us firstly assume that the volatility surface is being built form European option prices. Let us also consider a butterfly strategy where we are:

1: long a call with strike K - ΔK

2: long a call with strike K + ΔK

3: short two calls with strike K

The value of the strategy, B_0 at t = 0 is the following:

$$B_0 = e^{-rT} f(K,T)(\Delta K)^2 \tag{5}$$

where f(K,T) is the Probability Density Function of S_T at strike price K:

$$f(K,T) \approx e^{rT} \frac{C(K-\Delta K,T) - 2C(K,T)}{(\Delta K)^2} + \frac{C(K+\Delta K,T)}{(\Delta K)^2} \tag{6}$$

and if ΔK goes towards 0, we have that

$$f(K,T) = erT \frac{\partial^2 C}{\partial K^2} \tag{7}$$

"The volatility surface therefore gives the marginal risk-neutral distribution of the stock price, S_T, for any time, T"[35]. However, no information is given regarding the the joint distribution of the stock price at various periods T_1, ..., T_n. This result makes sense as the implied volatility surface is built from European option prices and these prices depend exclusively on the various marginal distributions of S_T[36].

1.8 Criteria for an effective representation of the implied volatility surface

Three criteria are of fundamental importance if we want to represent properly the volatility surface:

1: Parsimony - the representation must have the minimum amount of information that is requested to have the entire implied volatility surface for all strike prices and times to maturity.

2: Consistency - the information included in the representation is constantly built along the times to maturity and strikes, so that interpolation or extrapolation of missing points is possible.

3: Intuitiveness - the information gives the user with an understandable view about the shape of the implied volatility surface, and "each piece of the information distinctly affects one specific trait of the volatility surface"[37]

With the first criteria what is meant is that the representation is parsimonious if one can set up an interpolation-extrapolation scheme for both K and T, hence requiring only very few points as inputs. In theory, it should be hard to achieve, because for each expiry, a volatility smile has as many degrees of freedom as strikes given. But empirically volatilities do not move independently of each other, and it can be assumed that there are only three degrees of freedom:

[31] J. C. Hull - *Options, Futures, and other Derivatives Eighth Edition. (2012)*

[32] Ibidem

[33] Ibidem

[34] http://www-2.rotman.utoronto.ca/ hull/downloadablepublications/DHSPaperdraft7.pdf

[35] http://www.columbia.edu/mh2078/BlackScholesCtsTime.pdf

[36] Ibidem

[37] http://www.iasonltd.com/wp-content/uploads/2013/02/2b.pdf

level, slope and convexity. “In fact, as a principal component analysis can show, most of shape variations can be explained either by a parallel shift of the smile or by a tilt to the right or to the left or by a relative change of the wings with respect to the central strike”[38]. As a consequence, the minimum number of points needed at each time to maturity to portray the stylised movements is three, and they would consist in the volatilities for at-the-money options, out-of-the-money calls and out-of-the-money puts.

Moreover, these strike triplets, one for each expiry, must also be chosen in such a way that the resulting representation is consistent. Let us think of a very simple volatility surface with only two expiries: one week and ten years. For both expiries, one of the three strikes to choose may be set equal to the current price of the underlying asset (at-the-money spot). This choice is reasonable but not necessarily the best one. In fact, it would be better to replace the two at-the-money spot values with the forward prices at the two expiries, which can be viewed as expected values of the future underlying asset under suitable measures (the corresponding forward risk adjusted measures).

Things can even be worse for the other two points, since a meaningful selection criterion likely leads to different values for the two expiries: two chosen strikes may convey a good amount of information regarding the smile for the one week expiry, but may be not so informative for the ten year expiry. In fact, what matters (under a probabilistic point of view) is the relative distance of a strike from the central one, possibly expressed in volatility units, which makes the chosen strikes, and their corresponding implied volatilities, comparable throughout the entire range of expiries. A meaningful distance measure is provided by the delta of an option (in absolute terms), since it is a common indicator used in the market and it has the same signalling power as the relative distance from the at-the-money point (in units of total standard deviation).

Finally, the representation is intuitive if directly expressed in terms of three qualitative features of the surface, instead of three implied volatilities. These features, already mentioned above, are the level, the steepness and the convexity of the smile for each maturity. The level is correctly measured by the at-the money volatility.

Regarding the sets of expiry dates, a fixed number of maturities expressed as a fraction of years or as number of X months to maturity is the most intuitive choice to portray the implied volatility surface. Indeed, readers have an easier time by comparing times to maturity in the matrix and understand more intuitively which volatility corresponds to which strike and time to maturity[39].

[38] Ibidem
[39] Ibidem

2 No arbitrage conditions for the implied volatility surface

We shall assume a dynamically complete market. If there is no arbitrage opportunity then there is an equivalent martingale measure characterised "by the risk neutral transition density of the underlying stochastic process denoted by $\Theta(S_t, T|S_t, t, r_{t,\tau}, \delta_{t,\tau})$"[40], where S_t is the price at time t, T = t + τ = expiry date, τ time to expiration, $r_{t,\tau}$ the risk-free rate and $\delta_{t,\tau}$ the dividend rate.

Moreover, we have that the valuation function of a European call is the following:

$$C(S_t, t, K, T, r_{t,\tau}, \delta_{t,\tau}) = e^{-r_{t,\tau}\tau} * \int_0^\infty max(S_T - K, 0) \Theta(S_t, T|S_t, t, r_{t,\tau}, \delta_{t,\tau}) dS_T \quad (8)$$

Since the price function of a call option is a decreasing and convex function, deriving with respect to K, and together with the fact that Θ is both always positive and integrable to one, one receives:

$$-\ e^{-r_{t,\tau}\tau} \ \leq \ \frac{\partial C(S_t, t, K, T, r_{t,\tau}, \delta_{t,\tau})}{\partial K} \ \leq \ 0 \quad (9)$$

which means that there is monotonicity.

2.1 No butterfly arbitrage

Convexity comes from differentiating to the second degree with respect to the strike price (Breeden and Litzenberger, 1978):

$$\frac{\partial^2 C(S_t, t, K, T, r_{t,\tau}, \delta_{t,\tau})}{\partial K} = e^{-r_{t,\tau}\tau}\Theta(S_t, T|S_t, t, r_{t,\tau}, \delta_{t,\tau}) \geq 0 \quad (10)$$

2.2 No calendar arbitrage

Let us also take into consideration that the call price is bounded by:

$$max(e^{-\delta_{t,\tau}} S_t - e{-}r_{t,\tau}\tau K, 0) \leq C(S_t, t, K, T, r_{t,\tau}, \delta_{t,\tau}) \leq e^{-\delta_{t,\tau}\tau} S_t \quad (11)$$

2.3 Other remarks

All these constraints lead to nonlinear conditions for an arbitrage-free volatility surface. With the assumption of a strike-dependent volatility function, differentiating twice the Black-Scholes formula yields:

$$\Theta(K, T|S_t, t) = e^{-\delta_{t,\tau}\tau} S_t \sqrt{\tau}\eta(\bar{d}_1) \left[\frac{1}{K^2\hat{\sigma}\tau} + \frac{2\hat{d}_1}{K\hat{\sigma}\sqrt{\tau}}\frac{\partial\hat{\sigma}}{\partial K} + \frac{\bar{d}_1\bar{d}_2}{\hat{\sigma}}\left(\frac{\partial\hat{\sigma}}{\partial K}\right)^2 + \frac{\partial^2\hat{\sigma}}{\partial K^2}\right] \quad (12)$$

"where η is the pdf of a standard normal value"[41]. If we are to adopt no-arbitrage conditions, then the implied volatility surface would need to assume that $\Theta(K, T|S_t, t) \geq 0$ throughout all strikes and maturities. When considering T, there is only one weak constraint, which consist in the prices of American calls for the same strikes to be non-decreasing. When there are no dividends involved, the same rule applies to European calls[42]. However, the term structure of an implied volatility surface may be downward sloping as well.

Kahale' (2004) shows that, by treating the zero-δ and zero-r case (even if his approach can be taken a step forward by having a deterministic, time-dependent r_t and a dividend yield δ_t "which are the typical assumptions within the local volatility framework"[43]), we have the following argument (proven by Fengler in his thesis): "assume the existence of a deterministic, time-dependent interest rate r_t and a deterministic, time-dependent dividend yield δ_t. If $v^2(k, \tau_i)$ (which stands for our total variance) is a strictly increasing function for $\tau_i = T_i - t$ and i = 1,2, there is no calendar arbitrage"[44].

So, an implied volatility surface gotten from convex call price functions that observe $C_t(K_2, T_2) > e^{-\int_{T_1}^{T_2}\delta_t dt} C_t(K_1, T_1)$ or that is exclusively increasing in the total variance, has no (calendar) arbitrage.

Haug adds that skews should not be too steep at any given maturity, because otherwise there could be butterfly arbitrages. Additionally, "the term structure of the implied volatility cannot be too inverted"[45], because otherwise there would be calendar spread arbitrages.

Let us now add Rebonato's conditions that make sure of the existence of a risk-neutral density:

1: Market Conditions -we are in a complete market, with no taxes or transaction costs, no bid-ask spreads, short-sales are possible.

2: Traded instruments - possibility to trade both the underlying and plain-vanilla calls and put options for every single strike and time to maturity. There are also exists deterministic bonds and their income is determined by the risk-free rate r, and the payoffs of

[40] M. R. Fengler - *Arbitrage-Free Smoothing of the Implied Volatility Surface. (2005)*

[41] Ibidem

[42] Ibidem

[43] Ibidem

[44] Ibidem

[45] M. Haug - *Black-Scholes and the Volatility Surface*

the options depend exclusively on the price evolution of the underlying asset until expiration[46].

The important equations to take into consideration in order to not have arbitrages are (9), (10) and (11).

Henstchel (2003) states that possible errors in data that can lead to arbitrage are the following: "bid-ask bounce, asynchronous pricing and finite quote precision in option prices"[47].

2.4 If there is arbitrage

Even if arbitrage situations exist, they need not be removed necessarily from the data, and moreover imposing the "monotonicity and convexity constraints introduces distortions in the observed process"[48].

However there are two situations where the data must be free of arbitrage: the first one is the "method of extraction of risk neutral densities"[49]. Here it is vital that the data contains no arbitrage possibility so that the density extraction is appropriate. The other situation occurs through the model known as the "local volatility" one (Dupire's smile model)[50].

Arbitrage situations (real or "pseudo" arbitrage, the last one being potentially caused by prices measured with errors and retrieved through interpolation/extrapolation) can be of major impact, "given the nonlinear transformations applied in calls prices"[51]. In density extractions in a risk-neutral world the arbitrage situations in option prices can lead to very bad characteristics of the retrieved densities, "given that risk neutral is related to differentiate the data two times, introducing large fluctuations. Arbitrage conditions can lead to the presence of negative probability points and multimodality in the extracted risk neutral density"[52].

In local volatility models, the arbitrages can condition badly the local volatility, since estimation of the local volatilities has the possibility to be based through the direct option quotations or in the potential volatility surface if there are no-arbitrage possibilities. "The presence of arbitrage also affects the stability properties of the numerical methods used in the resolution of partial differential equations present in local volatility equations"[53].

[46] M. Poletti Laurini - *Imposing no-arbitrage conditions in implied volatility surfaces using constrained smoothing splines (2007)*
[47] Ibidem
[48] Ibidem
[49] Ibidem
[50] Ibidem
[51] Ibidem
[52] Ibidem
[53] Ibidem

3 Stochastic volatility and local volatility

3.1 Stochastic volatility: The Valuation Equation

"A stochastic volatility model is a model where the volatility itself is a stochastic process"[54]. Stochastic volatility models are key to volatility surface modelling because they help explain why options with varying K and T have varying BS implied volatilities[55].

One popular model is the Heston model, where the asset price follows a geometric brownian motion and the volatility also follows a geometric brownian motion with mean reversion[56]. Remember that a geometric brownian motion is a stochastic process with the following properties:

1. B_t meaning the brownian motion and it has normal increments. B_t - B_s N $\sim$ (0, t-s).

2. The increments are independent. B_t - B_s is independent of the past, meaning that for 0 < u < s it is not dependent on B_u.

3. Paths are continuous. "B_t, t $\geq$ 0 are continuous functions of t.

A Geometric Brownian Motion is a stochastic process where the logarithm of the randomly varying quantity follows a Brownian Motion"[57].

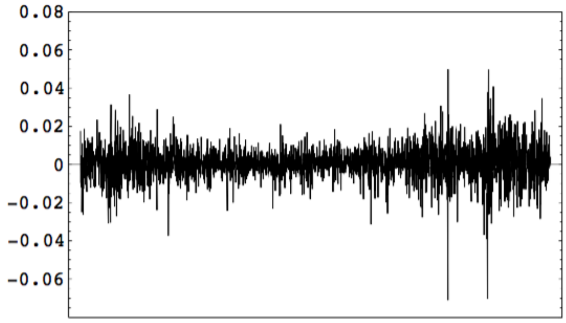
Other popular models include the Bates model (an extension of the Heston model, where in this case "the difference lies within the price process where a Poisson process is added"[58]) and the SVJJ model ("which has simultaneous jumps in the asset and the volatility"[59]). The problem with these models is that they are difficult to calibrate. Indeed, the Bates model has 9 parameters, and the SVJJ model many more.

However, simpler models that are easier to calibrate exist, such as the SABR model (where the volatility mean reversion property does not exist "and is therefore only good for short expirations theoretically"[60]) and the Stochastic Volatility Inspired model.

Stochastic volatility models also help when traders try to hedge: indeed, they have to constantly change the volatility assumption in order to make their prices equivalent to those of the market, and this process of restating the hedge ratios can become uncontrolled.

Because the distributions of assets are characterised by fat tails and leptokurtosis, the variance should be indeed modelled as a random variable. The concept of volatility clustering ("large moves following large moves and small moves following small moves"[61]) lets us conclude that volatility is auto-correlated, and that consequently there is mean reversion of volatility (an exception being jump-diffusion models[62]). Please refer to Figure 6 to see that indeed large moves follow other large moves and small moves follow previous small moves.

Figure 6: S&P Log returns 1990 to 1999 [63]



Why is volatility mean reverting? Consider the distribution of the volatilities of Apple in 100 years. If volatility was not mean revering, then the probability of volatility of Apple being between 1% and 100% would be low. Because we axiomatically believe that volatility would indeed lie in the [1%;100%] range, we can conclude that volatility is indeed mean reverting.

Following Wilmott as Gatheral does, suppose the following stochastic differential equations are satisfies by the price S and variance v in the following ways[64]:

$$dS_t = \mu_t S_t dt + \sqrt{v_t} S_t dZ_1 \tag{13}$$

$$dv_t = \alpha(S_t, v_t, t)dt + \eta\beta(S_t, v_t, t)\sqrt{v_t} dZ_2 \tag{14}$$

where μ_t, η and ρ are respectively the deterministic drift of stock price returns, the volatility of volatility ("vol of vol") and the correlation between random stock price returns and changes in v_t, and where dZ_1 and dZ_2 are the two Wiener processes[65]. Gatheral deduces that the valuation equation is the following:

$$\frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + \rho\eta v \beta S \frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2}\eta^2 v \beta^2 \frac{\partial^2 V}{\partial v^2} + rS\frac{\partial V}{\partial S} - rV = -(\alpha - \phi\beta\sqrt{v})\frac{\partial V}{\partial v} \tag{15}$$

where α is the drift function and β is the volatility function from the stochastic differential equation for

[54] E. Nilsson - *Calibration of the Volatility Surface. 2008*
[55] J. Gatheral - *The volatility surface: a practitioner's guide. (2006)*
[56] E. Nilsson - *Calibration of the Volatility Surface. 2008*
[57] http://www2.math.su.se/matstat/reports/seriec/2011/rep7/report.pdf
[58] E. Nilsson - *Calibration of the Volatility Surface. 2008*
[59] Ibidem
[60] Ibidem
[61] J. Gatheral - *The volatility surface: a practitioner's guide. (2006)*
[62] Ibidem
[63] http://www.cmap.polytechnique.fr/rama/teaching/ea2003/gatheral.pdf
[64] J. Gatheral - *The volatility surface: a practitioner's guide. (2006)*
[65] Ibidem

instantaneous variance[66]. It is always assumed that the stochastic differential equations for S and v are in risk-neutral terms because the aim is to fit models to option prices.

3.2 Local volatility

Local volatility is an average of instantaneous volatilities. The local volatility function σ_L(S,t) is consistent with the current prices of European options. Therefore a local volatility model treats volatility as a function of both the current asset price S_t and of time t. Such models are particularly of help when the underlying's volatility is a function mainly of the price of the asset. They are very useful in the creation of stochastic volatility models.

Since the "only source of randomness is the stock price, local volatility models are easy to calibrate. Also, they lead to complete markets where hedging can be based only on the underlying asset"[67]. With v_L representing the local variance $\sigma^2(S_0$, K, T), transforming to Black-Scholes implied volatility space gives the following final result:

$$v_L = \frac{\frac{\partial w}{\partial T}}{1 - \frac{y}{w}\frac{\partial w}{\partial y} + \frac{1}{4}\left(-\frac{1}{4} - \frac{1}{w} + \frac{y^2}{w^2}\right)\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{2}\frac{\partial^2 w}{\partial y^2}} \tag{16}$$

where w stands for the Black-Scholes implied total variance $w(S_0, K, T) = \sigma_B^2 S(S_0, K, T)T$, y for the log strike $ln\left(\frac{K}{F_T}\right)$, F_T is the forward price of the stock and T is the time to maturity.

Local volatility is used especially with exotic options. "There is no closed form formula for these contracts, therefore Monte Carlo simulation has to be used, and the preferable volatility is the local volatility"[68].

3.2.1 Dupire's equation

Given [69]

$$dS_t = rS_t dt + \sigma(t, S_t)S_t dW_t \tag{17}$$

where r stands for the risk-free rate (assumed to be bigger than 0), σ for the deterministic function and "W_t for the Brownian motion with respect to a risk-neutral probability measure"[70].

Let us consider the price of a European call option with T (time to maturity) bigger than 0 and K bigger than 0. Following what the risk-neutral valuation says, the formula is given by:

$$C_{T,K} = e^{-rT}\int_K^\infty (S-K)\phi_{T,S} dS \tag{18}$$

[66]Ibidem

[67]https://en.wikipedia.org/wiki/Local volatility

[68]E. Nilsson - *Calibration of the Volatility Surface. 2008*

[69]http://www2.math.uu.se/maciej/courses/PDE for Finance/DF2012.pdf

[70]Ibidem

where ϕ is the probability density function of S_t. Moreover,

$$\frac{\partial C}{\partial K} = -e^{-rT}\int_K^\infty \phi_{T,S} dS \tag{19}$$

and

$$\frac{\partial^2 C}{\partial K^2} = e^{-rT}\phi_{T,K} \tag{20}$$

and

$$\frac{\partial C}{\partial T} = -rC + e^{-rT}\int_K^\infty (S-K)\frac{\partial \phi}{\partial T}(T,S) dS \tag{21}$$

with the help of the Kolmogorov equation, we have the following formula, which is Dupire's formula:

$$\sigma_{T,K}^2 = 2\frac{\frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial K}}{K^2\frac{\partial^2 C}{\partial K^2}} \tag{22}$$

The right hand side of the previous equation can be computed from the known option prices, strikes and expirations. Consequently, Dupire's equation can be viewed as the local volatility function "regardless of what kind of process (stochastic volatility for example) actually governs the evolution of volatility"[71].

3.2.2 Problems with Dupire's equation

The equation firstly requires "continuous strikes and maturities"[72] (a way to partially solve the problem would be therefore through interpolation). Moreover, "numerical differentiation is very unstable"[73].

Lastly, another issue with the equation is that for K being far in-the-money or out-of-the-money the numerator or denominator may become extremely small, leading to unreal local volatilities.

[71]J. Gatheral - *The volatility surface: a practitioner's guide. (2006)*

[72]https://people.maths.ox.ac.uk/reisinge/Students/volaNotes.pdf

[73]Ibidem

4 The dynamics of the implied volatility

4.1 The risk-neutral drift formula

Assume the risk-neutral process followed by S:

$$\frac{dS}{S} = [r_t - q_t]dt + \sigma dz \tag{23}$$

where r_t, q_t, σ and z are respectively the risk-free rate, the dividend yield, the underlying's volatility and the Wiener process[74]. $\sigma_T K$(t, S) is the implied volatility at time t with strike K and maturity T and $V_T K$(t, S) is the implied variance of the option, so that $V_T K$(t, S) = $[\sigma_T K(t, S)]^2$. Suppose also that the process followed by $V_T K$ is the following:

$$dV_{TK} = \alpha_{TK}dt + V_T K \sum_{i=1}^{N} \theta_{TKi} dz_i \tag{24}$$

where z_i is the Wiener process of the volatility surface and θ_{TKi} is the sensitivity of $V_T K$ to the Wiener process, z_i. "The instantaneous volatility σ(t) is the limit of the implied volatility of an at-the-money option as its time to maturity approaches zero"[75]. Formally:

$$\lim_{T \to t} \sigma_{TK}(t, S) = \sigma(t) \tag{25}$$

where F is the forward price of the underlying. If c(S, $V_T K$, K, T) is the price of a European call option, we have the Black-Scholes-Merton formula:

$$c(S, V_{TK}, K, T) = e^{-\int_t^T q(\tau)d\tau} SN(d_1) - e^{-\int_t^T r(\tau)d\tau} KN(d_2) \tag{26}$$

where

$$d_1 = \frac{ln\left(\frac{S}{K}\right) + \int_t^T [r(\tau) - q(\tau)]d\tau}{\sqrt{V_{TK}(T-t)}} + \frac{1}{2}\sqrt{V_T K(T-t)} \tag{27}$$

and

$$d_2 = \frac{ln\left(\frac{S}{K}\right) + \int_t^T [r(\tau) - q(\tau)]d\tau}{\sqrt{V_{TK}(T-t)}} - \frac{1}{2}\sqrt{V_T K(T-t)} \tag{28}$$

Using Ito's lemma, Daglish, Hull and Suo arrive to the following equation, which also provides the no-arbitrage condition for the drift of the implied variance with respect to its volatility:

$$\alpha_{TK} = \frac{1}{T-t}(V_{TK} - \sigma^2) - \frac{V_{TK}(d1d2-1)}{4} * \sum_{i=1}^{N}(\theta_{TKi})^2 + \sigma d_2 \sqrt{\frac{V_{TK}}{T-t}} \sum_{i=1}^{N} \theta_{TKi}\rho_i \tag{29}$$

This equation provides the risk-neutral drift of the implied variance in with respect to its volatility. The last term is the drift that comes from the difference between the implied and instantaneous variances. The second term comes from the "part of the uncertainty about future volatility that is uncorrelated with the asset price. The first term arises from the correlation between the asset price and its volatility"[76].

4.2 Special cases

1^{st} case: V_{TK} is a deterministic function of t, T and K exclusively: the last equation can be rewritten as:

$$\alpha_{TK} = \frac{1}{T-t}(V_{TK} - \sigma^2) \tag{30}$$

or also

$$\sigma^2 = -\frac{d[(T-t)V_{TK}]}{dt} \tag{31}$$

which shows that σ is a deterministic function of t and T. The only model consistent with this first case is Merton's model. Also note that $V_{TK} = \sigma^2$ and the Black-Scholes-Merton model is obtained.

2^{nd} case: V_{TK} is independent of the underlying price, S. In this case, ρ_i is equal to 0 and equation (29) becomes the following:

$$\alpha_{TK} = \frac{1}{T-t}(V_{TK} - \sigma^2) - \frac{V_{TK}(d1d2-1)}{4}\sum_{i=1}^{N}(\theta_{TKi})^2 \tag{32}$$

"Both α_{TK} and θ_{TKi} must be independent of S. Because d_1 and d_2 depend on S we must have θ_{TKi} equal to 0 for all i"[77], which means that this second case reduces to the first case.

3^{rd} case: V_{TK} is a deterministic function ot t, T and K/S or K/F. Here:

$$V_{TK} = G\left(T, t, \frac{K}{F}\right) \tag{33}$$

where G is a deterministic function. The spot instantaneous volatility is given by

$$\sigma^2 = G(t, t, 1) \tag{34}$$

[74] http://www-2.rotman.utoronto.ca/ hull/ downloadablepublications/DHSPaperdraft7.pdf
[75] Ibidem
[76] Ibidem
[77] Ibidem

which is a deterministic function of t and T. This third case also reduces to the first case.

4^{th} case: V_{TK} is a deterministic function ot t, T, S and K. Here the model reduces itself to Dupire's equation.

5 Rules of thumb when creating the volatility surface

The rules of thumb for constructing the volatility surface fall in two categories. The first category is concerned with how the volatility surface varies throughout time. The usefulness is in the calculation of the greeks. The second category concerns with the relationship between "the volatility smiles for different option maturities at a point in time"[78]. Their advantage is that they help create a full volatility surface when market prices are available for a limited amount of options.

Three different rules have been considered here: the first two, the stick strike rule and the delta rule, belong to the first category of the rules of thumb. The third rule, also called the square root of time rule, falls into the second category of rules of thumb when constructing implied volatility surfaces. It consists in "filling in the blanks when a complete volatility surface is being produced"[79].

5.1 Sticky strike rule

This rule assumes that the implied volatility of the option does not depend on the underlying's price. Hence is assumes that the sensitivity of the option's price to the underlying's price S is $\frac{\partial c}{\partial S}$, where c is a function of S, t and $V_T K$ (which stands for the implied variance of the option). This assumption allows the Black-Scholes formulas to calculate the delta, using as volatility the option's implied volatility. The same can be claimed to be true for gamma[80].

The implied volatility of the option does not change for the entire life of the option. This is the basic version of the sticky strike rule, where the only model that is consistent with this type of version is the one "where the volatilities of all options are the same and constant"[81].

A more complex version of the sticky rule is where $V_T K$ is not dependent of S, but instead on other stochastic variables. In this case, the only version of the model that is consistent is "the model where the instantaneous volatility of the asset price is a function only of time"[82]. This is also called the Merton's model.

All versions of the sticky strike rule are not consistent with volatility smiles and skews. Indeed, if traders price options and other derivatives using various implied volatilities "and the volatilities are independent of the asset price, there must be arbitrage opportunities"[83].

[78]Ibidem
[79]Ibidem
[80]Ibidem
[81]Ibidem
[82]Ibidem
[83]Ibidem

Another version of the sticky delta rule used by trades is that $\sigma_T K(S,t) - \sigma_T F(S,t)$ is a function of K/F and T - t. "Here it is the excess of the volatility over the at-the-money volatility, rather than the volatility itself, which is assumed to be a deterministic function of the moneyness variable, K/F"[84]. This version of the sticky delta rule, also called the "relative sticky delta" model, allows the volatility level to vary throughout time to expiration and the configuration of the volatility term structure to vary, "but when measured relative to the at-the-money volatility, the volatility is dependent only on K/S and T - t"[85].

5.2 Sticky delta rule

This rule assumes that the implied volatility of the option depends on the variable K/S. Recall that the delta of a European option in stochastic volatility models is the following:

$$\Delta = \frac{\partial c}{\partial S} + \frac{\partial c}{\partial V_T K}\frac{\partial V_T K}{\partial S} \tag{35}$$

where the option price c is a function of S, t and $V_T K$. $\frac{\partial c}{\partial S}$ is the Black-Scholes delta with volatility being equal to implied volatility. Secondly, $\frac{\partial c}{\partial V_T K}$ is positive, therefore it follows that "if $V_T K$ is a declining (increasing) function of the strike price, it is an increasing (declining) function of S and Δ is greater then (less then) that given by Black-Scholes-Merton"[86]. Since for equities $V_T K$ is a decreasing function of K, BS understated the true delta. For FX options, instead, since they show a volatility smile, then the Black-Scholes "understates delta for low strike prices and overstates it for high strike prices"[87].

The most basic version of the stick delta rule is the one where IV is assumed to be a deterministic function of K/S and T - t. The only model that is consistent with this version is Merton's model where "instantaneous volatility of the asset price is a function only of time"[88]. Moreover, the model is inconsistent with volatility smiles and skews and for no arbitrage to exist, IVs must not depend on S and K.

A generalised version of the sticky delta rule is the "generalised sticky delta model", where the process $V_T K$ is dependent on K/S and T - t.

5.3 Square root of time rule

This rule gives a relationship between volatilities of options with various K and T at a given point in time.

[84]Ibidem
[85]Ibidem
[86]Ibidem
[87]Ibidem
[88]Ibidem

A version of the rule is:

$$\frac{\sigma_{TK}(S,t)}{\sigma_{TF}(S,t)} = \Phi\left(\frac{ln(\frac{K}{F})}{\sqrt{T-t}}\right) \tag{36}$$

"where Φ is a function, F is the forward price at time t of the underlying with a contract maturity of T"[89]. An alternative is the following formula:

$$\sigma_{TK}(S,t) - \sigma_{TF}(S,t) = \Phi\left(\frac{ln(\frac{K}{F})}{\sqrt{T-t}}\right) \tag{37}$$

If Φ does not change throughout time, then we are in a "stationary square root of time model". If it does change throughout time, then we are in a "stochastic square of root time model". If we know "the volatility smile for options that mature at T* and the at-the-money volatilities for other maturities"[90] then the complete volatility surface can indeed be computed. If F* is the forward price of the underlying for a contract maturing at T*, we can calculate the smile at time T through the result that $\sigma_{TK}(S,t) - \sigma_{TF}(S,t) = \sigma_{T*K*}(S,t) - \sigma_{T*F*}(S,t)$, where $K* = F*\left(\frac{K}{F}\right)^{\sqrt{(\frac{T*-t}{T-t})}}$.

If the volatility that is at-the-money is stochastic and independent of the underlying price S, the stationary square root of time model is a specific "case of the relative sticky strike model and the stochastic square root of time model is a particular case of the generalised sticky strike model"[91].

[89]Ibidem
[90]Ibidem
[91]Ibidem

6 Introduction to implied volatility surfaces in a Black-Scholes framework and more complex pricing models

6.1 Problems with the Black-Scholes model

The problem with the Black-Scholes implied volatility surface modeling is that the formula assumes that volatility is constant. Hence, we should have a flat implied volatility surface. Instead, as Figure 4 shows, the implied volatility surface is far from flat. In this graphical example we are looking at the implied volatility surface of a call option, which becomes very high when moneyness S/K becomes very large and time to maturity T approaches 0. Therefore, this assumption of constant volatility under the Black-Scholes-Merton model is violated. “With one σ input, the Black-Scholes-Merton model can only match one market quote at a specific date, strike and maturity”[92]. Thus, the implied volatilities at various K and T are different, and they do not define the term “return volatility”, but are closely related to volatility.

Also, “a particular weighted average of all IV^2 across different moneyness is very close to the expected return variance over the horizon of the option maturity”[93]. Lastly, as stated previously, implied volatility increases and reflects the time value of the option.

Moreover, returns are assumed to be lognormally distributed, which empirically is not true, as stated previously for FX, equity and index options. Skewness and kurtosis are not constant either. Thus, “a good option pricing model should account for return non-normality and its stochastic (time-varying) feature”[94].

Varying implementations of the Black-Scholes model will lead to varying implied volatility surfaces. If these implementations are correct expectations are that the volatility surfaces will be similar in construction and resemblance. Since single-stock options are usually American, call and put options will generally give rise to diverse implied volatility surfaces.

Even if the “Black-Scholes model is far from accurate, the language of Black-Scholes is pervasive. Every trading desk computes the Black-Scholes implied volatility surface and the greeks they compute and use are Black-Scholes greeks”[95]

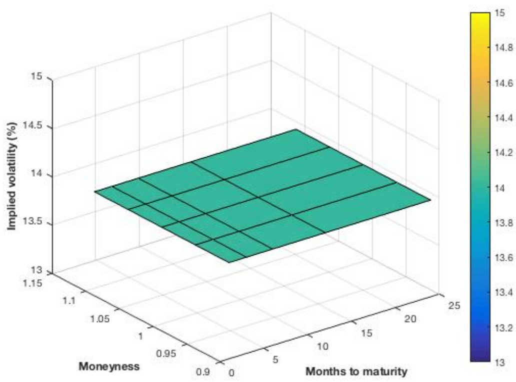
[92]http://faculty.baruch.cuny.edu/lwu/9797/Lec8.pdf
[93]Ibidem
[94]Ibidem
[95]http://www.columbia.edu/ mh2078/BlackScholesCtsTime.pdf

6.2 The implied volatility surface and the Black-Scholes model

It has to be remembered that an option value is characterised by two components: the intrinsic value (Max (S_0 - K; 0) for calls and Max(K - S_0; 0) for puts) and the time value (which is the optionality that the holder has to exercise the option throughout time. It is also defined as the price of the option subtracted by the option’s intrinsic value). The higher the volatility, the higher the option’s value, whether call or put, because of the fact that larger moves give the possibility for higher profits but yield no extra risk (because options give the right, but not the obligation, to buy or sell the underlying asset).

If the Black-Scholes-Merton assumptions hold, then we would have a flat implied volatility surface, as Figure 7 shows.

Figure 7: Flat volatility surface of Table 1



6.3 The implied volatility surface and implied binomial trees

The idea behind binomial pricing is given by the implied binomial tree, which helps understand the variations in implied volatility. This method is an adaptation of the Cox, Ross and Rubinstein method. The following is satisfied by an implied binomial tree:

“1: correct reproduction of the volatility smile;

2: node transition probabilities lying in [0,1]-interval only;

3: risk neutral branching process (forward price of the underlying asset equals the conditional expected value of itself) at each step”[96]

Conditions 2 and 3 guarantee also no-arbitrage. The objective of using the implied binomial tree is to estimate the implied probability distributions and hence the local volatility surfaces. Addiotionally, the implied

[96]http://edoc.hu-berlin.de/series/sfb-649-papers/2008-44/PDF/44.pdf

binomial tree may calculate the future stock price distributions through the Black-Scholes implied volatility surfaces, which are obtained from the European option prices[97].

6.3.1 How to construct the implied binomial tree

Given

$$\frac{dS_t}{S_t} = \mu(S_t, t)dt + \sigma(S_t, t, \cdot)dW_t \tag{38}$$

we can differentiate with respect to three different volatilities:

1: instantaneous volatility: $\sigma(S_t, t, \cdot)$

2: implied volatility: $\hat{\sigma}_t(K, T)$

3: local volatility: $\sigma_{K,T}(S_t, t)$

Therefore the implied binomial tree can be constructed "as a discretisation of an instantaneous volatility model"[98]:

$$\frac{dS_t}{S_t} = \mu(S_t, t)dt + \sigma(S_t, t)dW_t \tag{39}$$

"After the construction of the implied binomial tree, we are able to estimate a local volatility from underlying stock prices and transition probabilities"[99]. Only data that can be observed is used: therefore it is non-parametric naturally.

Through the Derman and Kani algorithm we can find the second moment of $log(S_{n+1})$ at $S_n = S_n^i$, which also the following implied volatility formula during Δt:

$$\sigma^2(S_n^i, \Delta t) = 2log\left(\frac{S_{n+1}^{i+1}}{S_{n+1}^i}\right)[p_{i+1}^n(1 - p_{i+1}^n)] \tag{40}$$

where

$$p_{i+1}^n = \frac{F_n^i - S_{n+1}^i}{S_{n+1}^{i+1} - S_{n+1}^i} \tag{41}$$

$$S_{n+1}^{i+1} = \frac{S_{n+1}^i[C(S_n^i, n\Delta t)e^{r\Delta t} - \rho_u] - \lambda_n^i S_n^i(F_n^i - S_{n+1}^i)}{C(S_n^i, n\Delta t)e^{r\Delta t} - \rho_u - \lambda_n^i(F_n^i - S_{n+1}^i)} \tag{42}$$

with

$$\rho_u = \sum_{j=i+1}^{n} \lambda_n^j(F_n^j - S_n^i) \tag{43}$$

$$S_{n+1}^i = \frac{S_{n+1}^{i+1}[e^{r\Delta t}P(S_n^i, n\Delta t) - \rho_l] - \lambda_n^i S_n^i(F_n^i - S_{n+1}^{i+1})}{e^{r\Delta t}P[S_n^i, (n+1)\Delta t] - \rho_l + \lambda_n^i(F_n^i - S_{n+1}^{i+1})} \tag{44}$$

with

$$\rho_l = \sum_{j=0}^{i-1} \lambda_n^j(S_n^i - F_n^j) \tag{45}$$

and

$$\lambda_{n+1}^0 = e^{-r\Delta t}[\lambda_n^0(1 - p_1^n)] \tag{46}$$

$$\lambda_{n+1}^{i+1} = e^{-r\Delta t}[\lambda_n^i p_{i+1}^n + \lambda_n^{i+1}(1 - p_{i+2}^n)], 0 \leq i \leq n-1 \tag{47}$$

$$\lambda_{n+1}^{n+1} = e^{-r\Delta t}[\lambda_n^n p_{n+1}^n] \tag{48}$$

F is the forward price given by the following formula:

$$F_n i = p_{i+1}^n S_{n+1}^{i+1} + (1 - p_{i+1}^n)S_{n+1}^i \tag{49}$$

and where $S_0^0 = 100$ and $\lambda_0^0 = 1$.

6.4 The implied volatility surface and Monte Carlo pricing

The Monte Carlo pricing method is a pricing method used to calculate the value of options "with multiple sources of uncertainty or with complicated features"[100]. It relies on the risk-neutral valuation and the price of the option is its discounted value. The application of this method is to firstly generate a large number of possible outcomes for the underlying price, and then calculate the "payoff" of the option for each path.

The idea behind the Monte Carlo pricing method is to calculate the price of the option given various inputs and then to calculate the implied volatility of each respective option through the given volatility model, be it non-stochastic, stochastic or local. For future reference to the rest of this research paper, the Monte Carlo pricing will use the Black-Scholes pricing method to calculate the implied volatility for each given option price.

The advantage of such method consists in the number of computations that can be computed and the speed with which they can be attained. Moreover, the user can decide how complex of a model he/she wants the Monte Carlo one to be. However, assumptions need to be reasonable, since the output relies totally on these assumptions (inputs). Moreover, the Monte Carlo method tends to underestimate the possibility of market crashes.

The reasoning behind choosing the Black-Scholes model for the Monte Carlo pricing is to compare itself to the Heston model, the SABR model and to the given market implied volatility and option prices. This way there can be a direct evaluation of which method is more precise for both pricing options and representing implied volatilities.

[97] Ibidem

[98] Ibidem

[99] Ibidem

[100] https://en.wikipedia.org/wiki/Monte Carlo methods for option pricing

7 Construction methodologies and parameterization for the implied volatility surface

7.1 Volatility Surface based on local stochastic volatility models

7.1.1 Heston model

The Heston model is the most well-known model of all the stochastic volatility models. This model is so popular mainly because it is relatively cheap in its computations. Even if it must be remembered that the Heston model is not realistic, if the inputs are chosen accurately, through all stochastic volatility models we can construct similar shapes of implied volatility surface and have a rough valuation of non-vanilla derivatives "in the sense that they are all models of the joint process of the stock price and instantaneous variance"[101].

The main aspect of this model is "the existence of a fast and easily implemented quasi-closed form solution for European options"[102].

The variance follows a Cox Ingersoll Ross (CIR) process. The Heston model consists in choosing $\alpha(S, v_t, t) = -\lambda(v_t - \bar{v})$ and $\beta(S, v, t) = 1$. We therefore have

$$dS_t = \mu_t S_t dt + \sqrt{v_t} S_t dZ_1 \tag{50}$$

and

$$dv_t = -\lambda(v_t - \bar{v})dt + \eta\sqrt{v_t}dZ_2 \tag{51}$$

with

$$(dZ_1 dZ_2) = \rho dt \tag{52}$$

"where λ is the speed of reversion of v_t to its long-term mean $\bar{v}$"[103]. Substituting the values of $\alpha(S, v_t, t)$ and of $\beta(S, v, t)$, we have the following:

$$\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \rho\eta v S\frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2}\eta^2 v\frac{\partial^2 V}{\partial v^2} + rS\frac{\partial V}{\partial S} - rV = \lambda(v - \bar{v})\frac{\partial V}{\partial v} \tag{53}$$

"In Heston's original paper, the price of risk is assumed to be linear in the instantaneous variance v in order to retain the form of the equation under the transformation from the statistical (or real) measure to the risk-neutral measure"[104].

There are various solutions to the Heston process:

1: The Eurler discretisation of the variance process is the following:

$$v_{i+1} = v_i - \lambda(v_i - \bar{v})\Delta t + \eta\sqrt{v_i}\sqrt{\Delta t}Z \tag{54}$$

with Z following a normal distribution with mean 0 and variance equal to 1. However this process may give a negative variance as a result. Should this occur, practitioners either adopt the absorbing assumption (if v < 0 then v = 0), or the "reflecting" assumption (if v < 0 then v = -v)[105].

2: If the Milstein discretisation is used, then, by going to 2 orders in the Ito-Taylor expansion of v(t + Δt), the following equation consists in the discretisation of the variance process:

$$v_{i+1} = v_i - \lambda(v_i - \bar{v})\Delta t + \eta\sqrt{v_i}\sqrt{\Delta t}Z + \frac{\eta^2}{4}\Delta t(Z^2 - 1) \tag{55}$$

Through this new equation, "the frequency with which the process goes negative is substantially reduced relative to the Euler case"[106]. It is not more computationally expensive to use the Milstein method, and it is to be preferred to the Euler method because it results fewer times in a negative variance.

3: using the Alfonsi (2005) discretization, we get the following equation:

$$v_{i+1} = v_i - \lambda(v_i - \bar{v})\Delta t + \eta\sqrt{v_i}\sqrt{\Delta t}Z - \frac{\eta}{2}\Delta t \tag{56}$$

4: Broadie and Kaya show "how to sample from the exact transition law of the process"[107]:

$$S_t = S_0 exp\left\{ -\frac{1}{2}\int_0^t v_s ds + \rho\int_0^t \sqrt{v_s}dZ_s + \sqrt{1-\rho^2}\int_0^t \sqrt{v_s}dZ_s^{\perp} \right\} \tag{57}$$

and

$$v_t = v_0 + \lambda\bar{v}t - \lambda\int_0^t v_s ds + \eta\int_0^t \sqrt{v_s}dZ_s \tag{58}$$

7.1.2 The implied volatility surface based on the Heston model

Let us firstly state that a model is useful if and only if it returns the current prices of European options. "That implies that we need to fit the parameters of our model (whether stochastic or local volatility model) to market implied volatilities"[108]. The best way to calibrate a model is if a method for computing the prices of options is fast, accurate and a function of the model parameters. Heston is such a favourable

[101] J. Gatheral - *The volatility surface: a practitioner's guide. (2006)*
[102] Ibidem
[103] Ibidem
[104] Ibidem
[105] Ibidem
[106] Ibidem
[107] Ibidem
[108] Ibidem

model. For local volatility models, numerical methods are required.

Let us firstly define the Black-Scholes gamma:

$$\Gamma_{BS}(S_t, \bar{\sigma(t)}) := \frac{\partial^2}{\partial S_t^2} C_{BS}(S_t, K, \bar{\sigma(t)}, T-t) \quad (59)$$

Gatheral explains with the following formula that $\bar{\sigma_0}$ is the BS implied volatility today for the option with strike K and expiration T:

$$\sigma_{BS}(K,T)^2 = \bar{\sigma_0}^2 = \frac{1}{T}\int_0^T \frac{\mathbb{E}\sigma_t^2 S_t^2 \Gamma_{BS}(S_t)|F_0}{\mathbb{E}S_t^2\Gamma_{BS}(S_t)|F_0} dt \quad (60)$$

where $\mathbb{E}$ stands for the expectation value.

With $x_t := log\left(\frac{S_t}{K}\right)$ and with $\hat{w_t} := \int_0^T ds \hat{v_s}$ being the expected total variance to time t, and letting $u_t := \mathbb{E}[v_t|x_T]$, $\lambda' = \lambda - \rho\eta/2$, $\bar{v}' = \bar{v}\lambda/\lambda'$ and $\hat{v_s}' := (v-\bar{v}')e^{-\lambda' s} + \bar{v}'$, Gatheral portrays to the reader the following equation for u_T:

$$u_T \approx \hat{v_T}' + \rho\eta\frac{x_T}{\hat{w_T}}\int_0^T \hat{v_s}e^{-\lambda'(T-s)}ds \quad (61)$$

This equation "gives us an approximate but surprisingly accurate formula for local variance within the Heston model (an extremely accurate approximation when $\rho = \pm 1$)"[109].

7.1.3 The Heston-Nandi model

After having seen the derivation in the previous section, we can observe that, if ρ = -1, the Heston process is written as the following:

$$dx = -\frac{v}{2}dt + \sqrt{v}dZ \quad (62)$$

and

$$dv = -\lambda(v-\bar{v})dt - \eta\sqrt{v}dZ \quad (63)$$

This choice of ρ = -1 was originally thought from Heston and Nandi "as the preference-free continuous time limit of a discrete GARCH option pricing model previously introduced by them"[110]. With only one source of randomness, there is the possibility to delete all volatility risk through delta hedging with the underlying, and hence "there is no volatility risk premium in this case"[111].

Let us rewrite the stochastic differential equation for v as the following:

$$dv = -\lambda'(v-\bar{v})dt - \eta dx \quad (64)$$

[109]Ibidem
[110]Ibidem
[111]Ibidem

with $\lambda' = \lambda + \eta/2, \bar{v}' = \bar{v}\lambda/\lambda'$, we note that the local variance will never be negative. From rewriting the second-last equation of the previous section as

$$\sigma_{BS}(K,T)^2 \approx \frac{1}{T}\int_0^T v_L(\tilde{x}_t)dt \quad (65)$$

we have that the local variance in the case ρ = -1 is given by:

$$v_{loc}(x_T,T) = (v-\bar{v}')e^{-\lambda' T} + \bar{v}' - \eta x_T\left[\frac{1-e^{1-\lambda' T}}{\lambda' T}\right] \quad (66)$$

"The whole expression must be bounded below by zero. All stock prices above the critical stock price at which the local variance reaches zero are unattainable"[112]. where x_t stands for $Log\frac{S}{S_0}$.

7.1.4 Comparing the Heston model to the Heston-Nandi model

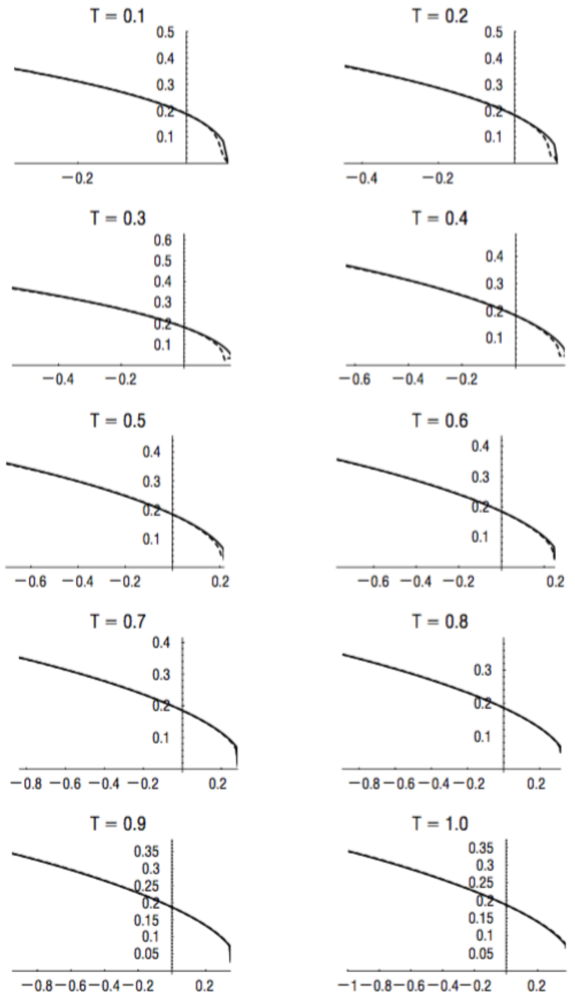
The following graphs show the a comparison of European implied volatilities "from the application of the Heston formula and from a numerical partial differential equation computation using the local volatilities given by the approximate formula. For each expiration T, the solid line is the numerical computation and the dashed line is the approximate formula"[113]. We can see that the stochastic and local volatility models price European options pretty much identically. Therefore, there exists a set of market parameters that permits to distinguish the effects of stochastic and local volatility assumptions on the valuation of multiple claims, "confident that European options are almost identically priced under both sets of assumptions"[114]. So, whenever there is a difference in the results of the two models, such difference can reasonably be attributed to the difference in dynamical assumptions rather than to the choice of parameters.

Of course, there will be more evident differences when we try to price exotic options, because of the discontinuity of the payoffs that they show.

In conclusion, "to value an option, it's not enough just to fit all the European option prices, we also need to assume some specific dynamics for the underlying"[115].

[112]Ibidem
[113]Ibidem
[114]Ibidem
[115]Ibidem

Figure 8: Comparison of Heston and Heston-Nandi models[116]



7.1.5 SABR model

The SABR model is also called the "stochastic alpha beta rho" model. It has been developed by Hagan and it has the following dynamics:

$$d\hat{F} = \hat{\alpha}\hat{F}^{\beta}dW_1 \tag{67}$$

and

$$d\hat{\alpha} = v\hat{\alpha}dW_2 \tag{68}$$

where the initial conditions are the following: $\hat{F}(0) = f$, $\hat{\alpha}(0) = \alpha$, and "the movements in the underlying forward are correlated with the movements in the underlying volatility"[117]:

$$dW_1 dW_2 = \rho dt \tag{69}$$

whereas the Brownian motions are correlated (ρ being different from 0). The parameters are:

"α: the initial variance
v: the volatility of volatility
β: the exponent for the forward rate
ρ: the correlation between the two Brownian motions"[118].

As stated previously in this research paper, volatility is not mean reverting in the SABR model. Therefore this model works only for short term maturities. "Nevertheless the model has the virtue of having an exact expression for the implied volatility smile in the short-expiration limit $\tau \to 0$. The resulting functional form can be used to fit observed short-dated implied volatilities and the model parameters α, β and ρ thereby extracted"[119].

The SABR model is usually used to model any type of forward rate. Being an extension to Black's model, the model does not derive option prices. "It instead produces estimations of the implied volatility curve, which are used as inputs in Black's model"[120] in order to find the potential option prices.

There are 3 particular cases to consider with respect to β:

1: when it equals 0, the resulting model is the stochastic normal model;

2: when it equals 1, the resulting model is the stochastic lognormal model;

3: when it equals $\frac{1}{2}$, the resulting model is the CIR model.

An advantage of the SABR model is that it is a very simple model which is "homogenous in f_t and α_t and has an approximating direct formula for the price of a European option"[121]. Its accuracy in the short-term to find the implied volatility curve is outstanding, which makes the SABR model effective to manage the smile risk "in markets where each asset only has a single exercise date, including swaptions and caplet/floorlet markets"[122].

A problem, thus, of the SABR model is that it is not able to fit the volatility surface on an asset which has various European options at various maturities, such as equity options. A partial solution to this problem has been discovered by Hagan with the dynamic SABR model. Even without this model, interpolation can be used to create the volatility surface.

The advantage of the SABR model is that it offers analytical formulas, which are compliant to Black's model to calibrate the parameters for the implied volatilities"[123].

Let us firstly revise the Black model by observing its pricing formulas:

$$C^{BS}(f, K, \sigma_B, T) = e^{-rT}(fN(d_1) - KN(d_2)) \tag{70}$$

[116] Ibidem
[117] http://www.fam.tuwien.ac.at/sgerhold/pub files/sem12/s sibetz nowak.pdf

[118] Ibidem
[119] Ibidem
[120] Ibidem
[121] Ibidem
[122] Ibidem
[123] Ibidem

with

$$d_{1,2} = \frac{ln\left(\frac{f}{K}\right) \pm \frac{1}{2}\sigma_B^2 T}{\sigma_B\sqrt{T}} \tag{71}$$

the implied volatility of the SABR model is given by the following formula:

$$\begin{aligned}\sigma_B(K,f) \\ = \frac{\alpha}{(fK)^{(1-\beta)/2}\left\{1+\frac{(1-\beta)^2}{24}ln^2\left(\frac{f}{K}\right)+\frac{(1-\beta)^4}{1920}ln^4\left(\frac{f}{K}\right)\right\}} \\ *\left(\frac{z}{x(z)}\right) \\ *\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fk)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta v\alpha}{(fK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}v^2\right]T\right\}\end{aligned} \tag{72}$$

with z being equal to

$$\frac{v}{\alpha}(fK)^{(1-\beta)/2}ln\left(\frac{f}{K}\right) \tag{73}$$

and x(z) being equal to

$$ln\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\} \tag{74}$$

With α, β and v being finally found, the implied volatility surface will be a function only of the forward price and the strike price. “This is the result of the fact that the SABR only produces implied volatilities for single maturities, the dependence of σ_B on T is not reflected in the notation σ_B (K, f)”[124].

Why choosing SABR over other possible models?

1: The underlying model makes assumptions as to the evolutions of the forward price and of the volatility, resulting in robust implied distributions;

2: The inputs of the model relate directly to market volatilities that can be observed in a simple Bloomberg Terminal or even in Yahoo! Finance;

3: Because of the two previous advantages, non-arbitrage restrictions, such as “the monotonicity of the first derivative of call option value with respect to the strike and the positivity of the second derivative, are handy to control”[125].

7.1.6 Local volatility with respect to implied volatility in the SABR model

Using implied volatility to calculate the local volatility gives a more accurate result. Additionally, massive amounts of time can be saved when computing the local volatilities. As Rebonato stated, “Implied volatility is the wrong number to put into wrong formula to obtain the correct price”[126]. Local volatility instead is consistent. It is a function which gives, through the Black-Scholes-Merton model, prices that are in line with those of the market.[127]

Given the previous implied volatility formula for the SABR model, let us recall the local volatility formula obtained by Dupire. The local volatility given in terms of implied volatility is the following[128]:

$$\begin{aligned}\sigma_{loc}(K,T) = \left(\frac{\sigma_{imp}}{T-t}+2\frac{\partial\sigma_{imp}}{\partial T}\right) \\ \div\left[K^2\left(\frac{\partial^2\sigma_{imp}}{\partial K^2}-d_1\sqrt{T-t}\left(\frac{\partial\sigma_{imp}}{\partial K}\right)^2\right.\right. \\ \left.\left.+\frac{1}{\sigma_{imp}}\left(\frac{1}{K\sqrt{T-t}}+d_1\frac{\partial\sigma_{imp}}{\partial K}^2\right)\right)\right]\end{aligned} \tag{75}$$

where

$$d_{1,2} \ = \ d_{+,-} \ = \ \frac{ln(F(T)/K) \pm \frac{1}{2}\sigma_{imp}^2(T-t)}{\sigma_{imp}\sqrt{T-t}} \tag{76}$$

7.1.7 Dynamic SABR model

A disadvantage of the static SABR model appears when data for options with diverse maturities is analysed. Indeed, large estimation errors can appear. “In order to overcome this problem, the following dynamic SABR model allows time dependency in some parameters”[129].

$$dF_t = \alpha_t F_t^\beta dW_t^1, F_0 = \hat{f} \tag{77}$$

$$d\alpha_t = v(t)\alpha_t dW_t^2, \alpha_0 = \alpha \tag{78}$$

with ρ also time dependent. The dynamic SABR model results in the following equation to approximate the implied volatility:

$$\begin{aligned}\sigma_{model}(K,\hat{f},T) = \frac{1}{w}\left(1+A_1(T)ln\left(\frac{K}{\hat{f}}\right)\right. \\ \left.+A_2(T)ln^2\left(\frac{K}{\hat{f}}\right)+B(T)T\right)\end{aligned} \tag{79}$$

where

$$A_1(T) = \frac{\beta-1}{2}+\frac{\eta_1(T)w}{2} \tag{80}$$

$$\begin{aligned}A_2(T) = \frac{1-\beta}{12}+\frac{1-\beta-\eta_1(T)w}{4} \\ +\frac{4v_1^2+3(\eta_2^2(T)-3\eta_1^2(T))}{24}w^2\end{aligned} \tag{81}$$

[124]Ibidem

[125]http://kodu.ut.ee/ spartak/papers/sabr.pdf

[126]http://www.performancetrading.it/Documents/ KscComparison/Ksc Local.htm

[127]Ibidem

[128]http://kodu.ut.ee/ spartak/papers/sabr.pdf

[129]https://www.researchgate.net/publication/257200550 Static and dynamic SABR stochastic volatility models Calibration and option pricing using GPUs

$$B(T) = \frac{1}{w^2}\left(\frac{(1-\beta)^2}{24} + \frac{w\beta\eta_1(T)}{4} + \frac{2v_2^2(T) - 3\eta_2^2(T)}{24}w^2\right) \tag{82}$$

with

$$v_1^2(T) = \frac{3}{T^3}\int_0^T (T - t)^2 v^2(t)dt \tag{83}$$

$$v_2^2(T) = \frac{6}{T^3}\int_0^T (T - t)tv^2(t)dt \tag{84}$$

$$\eta_1(T) = \frac{2}{T^2}\int_0^T (T - t)v(t)\rho(t)dt \tag{85}$$

$$\eta_2^2(T) = \frac{12}{T^4}\int_0^T\int_0^t\left(\int_0^s v(u)\rho(u)du\right)^2 dsdt \tag{86}$$

If v $= v_0$ and $\rho = \rho_0$, then $v_1(T) = v_2(T) = v_0$, $\eta_1(T) = \eta_2(T) = v_0\rho_0$ and the dynamic SABR model becomes the static SABR model.

7.1.8 SABR model with jumps

Even if adding jumps in volatility makes any model more realistic, they do not add any value in terms of shaping the volatility surface for very short times to maturity[130]. Nevertheless, it is important to see how Medveded and Scaillet have derived their formula for short-dated implied volatilities.

Let us firstly consider the stochastic volatility model with jumps:

$$\frac{dS_t}{S_t} = \sigma_t dZ_1 + J(\sigma_t)dq_t \tag{87}$$

and

$$d\sigma_t = a(\sigma_t)dt + b(\sigma_t)dZ2 \tag{88}$$

"where the term dq is a standard Poisson process with intensity $\lambda_J(\sigma_t)$ and $J(\sigma_t)$ is a $(-1,\infty)$-valued random variable with density f samples at each jump"[131]. The jump compensator μ_J is equal to:

$$\lambda_J\int_{-1}^{\infty} f(x)dx \tag{89}$$

[130]J. Gatheral - *The volatility surface: a practitioner's guide. (2006)*
[131]Ibidem

The short-dated implied volatility is given by the following:

$$I(z,\tau,\sigma) = \sigma + \tilde{I}_1(z;\sigma)\sqrt{\tau} + \tilde{I}_2(z;\sigma)\tau + O(\tau\sqrt{\tau}) \tag{90}$$

where $\tilde{I}_1$ and $\tilde{I}_2$ are the following:

$$\tilde{I}_1(z;\sigma) = I_1(z;\sigma) - \mu_J g(z) + \eta_J h(z) \tag{91}$$

$$\begin{aligned}\tilde{I}_2(z;\sigma) &= I_2(z;\sigma) + \frac{1}{2\sigma}(\mu_J g(z) - \eta_J h(z))^2 z^2 \\ &-\left\{-\frac{\mu_J\sigma}{2} - \sigma\lambda_J + \frac{\mu^2}{\sigma} + \frac{\mu_J b(\sigma)\rho}{2\sigma}\right\}g(z)z \\ &-\left\{\frac{\eta_J\sigma}{2} + \sigma\chi_J - \frac{\mu_J\eta_J}{\sigma} - \frac{\eta_J b(\sigma)\rho}{2\sigma}\right\}h(z)z \\ &+\frac{\rho b(\sigma)\mu_J}{2\sigma} - \frac{\rho\partial_\sigma b(\sigma)\mu_J}{2} + \frac{\mu_J^2}{2\sigma} - \frac{\sigma\mu_J}{2} - \lambda_J\sigma\end{aligned} \tag{92}$$

"where $\mu_J = \lambda_J\int_0^\infty xf(x)dx$, $\chi = \lambda_J\int_0^\infty xf(x)dx$ are respectively the positive part of the jump compensator and the probability of an upwards jump and $g(z) = \frac{N(-z)}{N'(z)}$; $h(z) = \frac{1}{N'(z)}$"[132].

Every single term that has a subscript J is jump-related, and disappears in the absence of jumps.

7.1.9 Local stochastic volatility model

The local stochastic volatility model is a hybrid of various local and stochastic volatility models. Here we represent the example of a hybrid Heston model with a local volatility model (first of two examples), with the following dynamics:

$$df^{LSV}(t) = \sigma(f^{LSV}(t),t)\sqrt{v(t)}f^{LSV}dW_1(t) \tag{93}$$

$$dv(t) = k(\theta - v(t))dt + epsilon\sqrt{v(t)}dW_2(t) \tag{94}$$

where f stands for the forward price at time t and v for the variance.

To calibrate the model, firstly calibrate the stochastic volatility part, and then adopt a local stochastic volatility correction. This 2-step process is valid because of "the observation that the forward skew dynamics in stochastic volatility setting are mainly preserved under the local stochastic volatility correction"[133]

Let us firstly compute the conditional expected variance v(t) given f^{LSV}(t):

$$E[v(t)|f^{LSV}(t) = f] = \frac{\int_0^\infty vp(t,f,v)dv}{\int_0^\infty p(t,f,v)dv} \tag{95}$$

[132]Ibidem
[133]http://arxiv.org/pdf/1107.1834.pdf

The second step is to adjust σ following Gyongy's identity set up for the local volatilities outputted by the local stochastic volatility function:

$$(\sigma_{LV}^{LSV})^2(f,t) = \sigma^2(f,t)E[v(t)|f^{LSV}(t) = f] = (\sigma_{LV}^{Market})^2(f,t) \quad (96)$$

Lastly, the two steps must be repeated until $\sigma(f,t)$ has converged (usually 1 or two trials are more than sufficient).

The second way to compute local volatilities is through local volatility ratios through the following procedure:

1: Use Gyongy's formula to calculate the local volatilities of the local stochastic volatility and stochastic volatility models:

$$(\sigma_{LV}^{LSV})^2(f,t) = \sigma^2(f,t)E[v(t)|f^{LSV}(t) = f] = (\sigma_{LV}^{Market})^2(f,t) \quad (97)$$

and

$$(\sigma_{LV}^{SV})^2(x,t) = E[v(t)|f^{SV}(t) = x] \quad (98)$$

2: Take the ratio of the two volatilities calculated in 1:

$$\sigma(t,f) = \frac{\sigma_{LV}^{Market}(f,t)}{\sigma_L V^{SV}(x,t)}\sqrt{\frac{E[v(t)|f^{SV}(t) = x]}{E[v(t)|f^{LSV}(t) = f]}} \approx usingx = H(f,t) \approx \frac{\sigma_{LV}^{Market}(f,t)}{\sigma_{LV}^{SV}(H(f,t),t)} \quad (99)$$

7.2 Volatility Surface based on Levy processes

7.2.1 Implied Levy volatility

The idea behind using a model based on Levy processes is to "better handle short term skews (observed especially in FX and commodity markets)"[134]. When we observe models with continuous paths like a diffusion model, the price behaves like a Brownian motion and the probability that the price moves greatly over very short periods is very small. Hence in these specific models the prices of small T out-of-the-money options are lower than those observed in the real markets. However, if there is the possibility for the price to jump, then the probability of high volatility in the price of the underlying in the very short term is non-negligible.

There are two categories of Levy processes:

[134]C. Humescu - *Implied volatility surface: construction methodologies and characteristics. (2011)*

1: jump diffusion processes: with jumps being evaluated as rare events, "in any given finite interval there are only finite many jumps;

2: infinite activity Levy processes: in any finite time interval there are infinitely many jumps"[135].

Options literature confirms that there is presence of jumps in the S&P options, for example[136].

Levy processes manage to show the volatility smile for a single time to maturity very well, but when we are talking in terms of various maturities, the calibration process is less precise. "To calibrate a jump-diffusion model to options of several maturities at the same time, the model must have a sufficient number of degrees of freedom to reproduce different term structures"[137].

The first "implied Levy volatilities" have been introduced through the implied Levy space volatility and the implied Levy time volatility where, instead of following a normal distribution, the distribution follows a process more similar to empirical observations.

This is the model being followed:

$$S(t) = S_0 exp[(r - q - w)t + \sigma X(t)] \quad (100)$$

where $\sigma > 0$, and r, q, w and X = (X(t), $t > 0$) are respectively the risk-free rate, the dividend yield, a term that is put in order to adjust the equation to being dynamic risk-neutral and a stochastic process that begins at zero and that "has stationary and independent increments distributed according to the newly selected distribution"[138].

Let us see as an example the implied Levy time volatility: we firstly start from a similar Levy process and we consider the following dynamics:

$$S(t) = S_0 exp[(r - q - w\sigma^2)t + X(t)] \quad (101)$$

and

$$w = log(\phi(-i)) \quad (102)$$

"The volatility parameter $\sigma = \sigma(K,T)$ needed to match the model price with a given market price is called the implied Levy space volatility"[139].

[135]Ibidem

[136]A. Medvedev and O. Scaillet - *Approximation and calibration of short-term implied volatilities under jump-diffusion stochastic volatility.* http://www.hec.unige.ch/www/hec/m1/RecherchePublications/CahiersRecherche/2006/simple Text/0/content files/file8/2006.01.01.01.01.01.01.01.01.01.01.01.01.pdf, 2006.

[137]C. Humescu - *Implied volatility surface: construction methodologies and characteristics. (2011)*

[138]Ibidem

[139]http://cermics.enpc.fr/cnf/Guillaume.pdf

7.2.2 Bates (SVJ) model

In the Bates model the implied volatilities for short maturities can be assumed to have been potentially affected by the presence of jumps, while "the smile for longer maturities and the term structure of implied volatility is taken into account using the stochastic volatility process"[140].

The idea behind the "Stochastic Volatility plus Jumps in the underlying only" model is that there is a combination between the stock price jumps and the stochastic volatility.

Using the Merton-style lognormally distributed jump in the Heston process we have the following[141]:

$$dS = \mu S dt + \sqrt{v} S dZ_1 + (e^{\alpha+\delta\epsilon} - 1) S dq \tag{103}$$

$$dv = -\lambda(v - \bar{v})dt + \eta\sqrt{v} dZ_2 \tag{104}$$

with $dZ_1\ dZ_2 = \rho$dt, $\epsilon \sim$ N(0,1) and the following poisson process: dq = 0 with probability 1 - $\lambda_J$dt and dq = 1 with probability $\lambda_J$dt, where λ_J is the hazard rate / jump intensity. By re-substituting in Gatheral's valuation equation, "the characteristic function for this process is just the product of Heston and jump characteristic functions"[142]:

$$\phi_T(u) = e^{C(u,T)\bar{v}+D(u,T)v} e^{\psi(u)T} \tag{105}$$

with

$$\psi(u) = -\lambda_J iu(e^{\alpha+\delta^2/2} - 1) + \lambda_J(e^{iu\alpha-u^2\delta^2/2} - 1) \tag{106}$$

$$C(u,T) = \lambda\left[r_-\tau - \frac{2}{\eta^2}log\left(\frac{1-ge^{-d\tau}}{1-g}\right)\right] \tag{107}$$

$$D(u,T) = r_-\frac{1-e^{-d\tau}}{1-ge^{-d\tau}} \tag{108}$$

where

$$g := \frac{r_+}{r_-} \tag{109}$$

and

$$r_\pm = \frac{\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\gamma} =: \frac{\beta \pm d}{\eta^2} \tag{110}$$

where

$$\alpha = -\frac{u^2}{2} - \frac{iu}{2} + iju \tag{111}$$

$$\beta = \lambda - \rho\eta j - \rho\eta iu \tag{112}$$

$$\gamma = \frac{\eta^2}{2} \tag{113}$$

where with the following two equations we get respectively the implied volatilities and the at-the-money volatility skew at any given expiration:

$$\int_0^\infty \frac{du}{u^2+\frac{1}{4}} Re\left[e^{-iuk}\left(\phi_T(u-i/2) - e^{-\frac{1}{2}(u^2+\frac{1}{4})\sigma_{BS}^2 T}\right)\right] = 0 \tag{114}$$

and

$$\left.\frac{\partial w_{BS}}{\partial k}\right|_{k=0} - e^{\frac{\sigma_{BS}^2 T}{8}}\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{T}} * \int_0^\infty du \frac{uIm[\phi_T(u-i/2)]}{u^2+\frac{1}{4}} \tag{115}$$

where the first of these last two equations gives the "relationship between the implied volatility surface and the characteristic function of the underlying stock process"[143].

One problem with the SVJ model is that because of the absence of correlation between the volatility process and the jump, after the jump has occurred the volatility will stay unchanged. Empirically and intuitively, "after a large move in the underlying, implied volatilities always increase substantially (i.e., they jump)"[144].

7.2.3 SVJJ model

To account for the unrealistic fact that instantaneous volatility would not jump after the stock price jumps, the SVJJ model has been adapted to such situation. Indeed, let us not forget that large moves follow large move and vice-versa. Hence, it makes sense intuitively and empirically that after a stock jump there is also a volatility jump.

The idea behind this model is that "jumps in the stock price are accompanied by a jump in the instantaneous volatility"[145]. The characteristic function in this case is the following:

$$\phi_T(u) = exp\{\hat{C}(u,T)\bar{v} + \hat{D}(u,T)v\} \tag{116}$$

with C(u,T) and D(u,T) being respectively:

$$\hat{C}(u,T) = C(u,T) + \lambda_J T[e^{iu\alpha-u^2\sigma^2/2}I(u,T) - 1 - iu(e^{\alpha+\delta^2/2} - 1)] \tag{117}$$

and

$$\hat{D}(u,T) = D(u,T) \tag{118}$$

[140] C. Humescu - *Implied volatility surface: construction methodologies and characteristics. (2011)*

[141] J. Gatheral - *The volatility surface: a practitioner's guide. (2006)*

[142] Ibidem

[143] Ibidem

[144] Ibidem

[145] Ibidem

where

$$I(u,T) = \frac{1}{T}\int_0^\infty e^{\gamma_v D(u,T)}dt$$
$$= -\frac{2\gamma_v}{p_+p_-}\int_0^{-\gamma_v D(u,T)} \frac{e^{-z}dz}{(1+z/p_+)(1+z/p_-)} \quad (119)$$

and

$$p_\pm = \frac{\gamma_v}{\eta^2}(\beta - \rho\eta ui \pm d) \quad (120)$$

Whenever γ_v goes towards 0, I(u,T) approaches 1 and we are back to the SVJ model. Moreover, when T approaches 0, I(u,T) approaches 1 and in this case the SVJJ characteristic function is the same as the SVJ one.

Gatheral shows in his book that the term structure of volatility skew corresponds to the intuition. Inserting volatility jumps does not contribute at all to explain the presence of potential very short-dated volatility skews. But when compared to stochastic volatility and SVJ models, "it does reduce the volatility of volatility required to fit longer-dated volatility skews even if that comes at the expense of a seemingly even more unreasonable estimate for the average stock price jump"[146].

Unfortunately Gatheral also shows that the SVJJ model has many more parameters and that it is harder to fit to observable market option prices. "The SVJ model thus emerges as a clear winner in comparison between Heston, SVJ and SVJJ models"[147].

7.3 Volatility Surface based on models for the dynamics of implied volatility

7.3.1 Carr and Wu approach (dynamic implied volatility)

The Carr Wu approach assumes that there is a model in between stochastic instantaneous volatility models and market models of implied volatilities. For the first type of these two models, the starting point is that S_0 is known and the level of financing is known too. Moreover, there are assumptions about the stock price and instantaneous return volatility dynamic. The implications are the level and shape of the implied volatility surface. In terms of calibration, "parameters governing the price/volatility dynamics and the initial volatility level can be calibrated to a finite number of option observations. The calibrated model can be used to construct the whole implied volatility surface"[148]. The disadvantages of such model are that the initial instantaneous volatility level cannot be observed and that it is a difficult model to calibrate.

Concerning the market models of implied volatilities, the starting situation is the following: "known initial option implied volatility level (on a single option, a curve, or over the whole surface)"[149]. In the assumptions we have the martingale component of the implied volatility dynamics, and the implications are the following: risk exposures and the drift on the implied volatility dynamics. The drawback is the following: "given an entire initial implied volatility surface, one is not free to choose any martingale component of dynamics"[150].

The Carr and Wu approach comes in between these two approaches with the following givens:

1: Starting point: S_0 and level of financing.

2: Assumptions: "Stock price and option implied volatility dynamics (both drift and diffusion), instead of instantaneous return volatility dynamics"[151].

3: Implications: the level of the implied volatility surface at a given date.

4: Calibration: "Parameters governing the implied volatility dynamics and the initial instantaneous volatility level can be calibrated to a finite number of vanilla option implied volatility observations"[152]. The model can eventually be adopted to build the entire implied volatility surface. The calibration is carried "directly from implied volatility dynamics to implied volatility surface. It is 100 times faster than calibrating standard option pricing models of similar complexities"[153].

There is the assumption of rates being equal to zero. Moreover, $\frac{dS_t}{S_t} = s_t dW_t$, where s_t is left unspecified. The specific option's implied volatility, called $I_t(K,T)$, follow the following process:

$$dI_t(K,T) = \mu_t dt + w_t dZ_t \quad (121)$$

for all K > 0 and T > t and where μ_t is the drift and w_t is the volatility of volatility and can both depend on K, T and I(K,T). There is also the assumption that "one Brownian motion Z_t drives the whole implied volatility surface"[154]. There is lastly the requirement "that no dynamic arbitrage be allowed between any option at (K,T) and a basis option at (K_0, T_0) and the stock"[155].

With $P_t(K,T)$ being the option value, the fundamental partial differential equation becomes the fol-

[146]Ibidem
[147]Ibidem
[148]Carr and Wu - *A new simple approach for constructing implied volatility surfaces. (2011)*
[149]Ibidem
[150]Ibidem
[151]Ibidem
[152]Ibidem
[153]Ibidem
[154]Ibidem
[155]Ibidem

lowing:

$$-B_t = \mu_t B_\sigma + \frac{s_t^2}{2} {S_t}^2 B_{SS} + \rho_t w_t s_t S_t B_{S\sigma} + \frac{w_t^2}{2} B_{\sigma\sigma} \tag{122}$$

where the partial differential equation defined relationship that is linear between the theta (B_t) of the option and its vega (B_σ), dollar gamma (${S_t}^2 B_{SS}$), dollar vanna ($S_t B_{S\sigma}$) and volga ($B_{\sigma\sigma}$).

This class of implied volatility surfaces defined by the partial differential equation has been redefined as the "Vega-Gamma-Vanna-Volga" model by Carr and Wu.

By plugging the partial derivatives of the BS formula, the partial differential equation reduces to the following quadratic equation:

$$\frac{I_t^2}{2} - \mu_t I_t \tau - \left[\frac{s_t^2}{2} - \rho_t w_t s_t \sqrt{\tau} d_2 + \frac{w_t^2}{2} d_1 d_2 \tau\right] = 0 \tag{123}$$

"If (μ_t, w_t) do not depend on $I_t(K,T)$, we can solve the whole implied volatility surface as the solution to a quadratic equation"[156].

The two have gone further to develop two variance dynamics: the "square root implied variance" one and the "log-normal implied variance" one. In the first one the implied volatility surface is represented in terms of the standardised moneyness $\frac{ln \frac{K}{S_t} + \frac{1}{2} I^2 \tau}{I\sqrt{\tau}}$ and the term τ = T - t.

The square-root implied variance dynamics formula is the following:

$$dI_t^2 = k[\theta - I_t^2]dt + 2we^{-\eta(T-t)} I_t dZ_t \tag{124}$$

and the implied volatility surface $v(z,\tau)$ solves the following quadratic equation:

$$\begin{aligned}(1+k\tau)v_t^2(z,\tau) - (w^2 e^{-2\eta\tau}\tau^{3/2}z)v_t(z,\tau) \\ - \big[(k\theta - w^2 e^{-2\eta\tau})\tau + s_t^2 + 2\rho w s_t e^{-\eta\tau} \\ \sqrt{\tau}z + w^2 e^{-2\eta\tau}\tau z^2\big] = 0\end{aligned} \tag{125}$$

where, in the limit $\tau = 0$, $v_t^2(z,0) = s_t^2$ and in the limit $\tau = \infty$, $v_t^2(z,\infty) = \theta$ and where the at-the-money implied variance (hence when z = 0) term structure is the following: $a_t^2(\tau) = \frac{(k\theta - w^2 e^{-2\eta\tau})\tau + s_t^2}{(1+k\tau)}$.

With the log-normal implied variance dynamics, instead, we have the following equation:

$$\begin{aligned}dI_t^2(K,T) = k[\theta - I_t^2(K,T)]dt \\ + 2we^{-\eta(T-t)} I_t^2(K,T) dZ_t\end{aligned} \tag{126}$$

where the implied variance surface $(\hat{I_t^2}(k,\tau))$ solves the following quadratic equation:

[156]Ibidem

$$\begin{aligned}&\frac{w^2}{4} e^{-2\eta\tau}\tau^2(\hat{I_t^4}(k,\tau)) \\ &\quad + [1 + k\tau + w^2 e^{-2\eta\tau}\tau - \rho s_t w e{-}\eta\tau\tau]\hat{I_t^2}(k,\tau) \\ &\quad - [s_t^2 + k\theta\tau + 2\rho s_t w e^{\eta\tau} k + w^2 e^{-2\eta\tau} k^2] = 0\end{aligned} \tag{127}$$

where in the limit of $\tau = 0$, $(\hat{I_t^2}(k,\tau)) = w^2k^2 + 2\rho s_t w k + s_t^2$, in the limit of $\tau = \infty$, $(\hat{I_t^2}(k,\tau)) = \theta$ and where the at-the-money implied variance (therefore when z = 0) term structure is the following: $a_t^2(\tau) = \frac{(k\theta\tau + s_t^2}{1+(k+w^2 e^{-2\eta\tau})}$.

The two models have six time-varying coefficient and have the advantage that "given time-t values on the six coefficients, the whole implied volatility surface at time t can be solved as the solution to quadratic equations"[157].

Lastly, according to Caar and Wu, their model is about 100 times faster and much more accurate than the Heston model. They have stated that options traders prefer to use the Black-Scholes-Merton implied volatilities. Indeed, "directly modeling implied volatility dynamics and generating direct implications on the implied volatility surface shape are both attractive ideas"[158]. What happens is that market models of implied volatilities try to model the implied volatility dynamics while taking the implied volatility surface as given. The implied volatility surface "can put severe constraints on what the former can be, or vice versa. We directly model the implied volatility dynamics, and we derive the dynamic-no-arbitrage implication on the shape of the implied volatility surface. The two are guaranteed to be consistent. Market deviations from model implications can serve as relative trading opportunities"[159].

Questions that arise with this method are the following:

1: How can we guarantee static no-arbitrage across different K and T and among many options?

2: How can we "link the implied volatility dynamics to the dynamics of the instantaneous return variance rate?

3: How can we accommodate multiple factors and discontinuous dynamics in both prices and implied volatilities?"[160]

[157]Ibidem
[158]Ibidem
[159]Ibidem
[160]Ibidem

7.4 Volatility Surface based on parametric representations

7.4.1 Polynomial parametrisation

This type of representation, suggested by Dumas, Fleming and Whaley[161], proposes the implied volatility surface as a function of the moneyness M $\triangleq ln(\frac{F}{K}/\sqrt{T})$, volatility would be described by the following equation:

$$\sigma(M,T) = b_1 + b_2 M + b_3 M^2 + b4T + b5MT \quad (128)$$

It must be stated that this model was thought mainly for the commodities markets (in particular, for the Oil markets), and the problem with this model is that is gives an "average" shape since it assumes that "the quadratic function of volatility versus moneyness is the same across all maturities"[162]. Indeed, "the increasing power of the polynomial volatility function does not offer a solution, since this volatility function will still be the same for all maturities"[163]. In order to solve these issues, semi parametric representations have been analysed further by Borovkova and Permana[164], where they have approximated the implied volatility surface by a "quadratic function which has time dependent coefficients"[165]. However, such parametrisations have various disadvantages:

1: They are not able to guarantee an arbitrage-free volatility surface;

2: The dynamics of the surface cannot realistically be observed.

Since these specific parametrisations are mainly useful in Oil markets and since the empirical part of this research paper will focus on the equity markets, there will be no further analysis on them.

7.4.2 Stochastic volatility inspired parametrisation

Stochastic volatility inspired parametrisation is applicable mainly in both the equity and the energy markets, and can be used along conditions for the "no vertical and horizontal spread arbitrages"[166]. The main characteristics of the stochastic volatility inspired parametrisation are the following:

1: "Each time slice of the implied volatility surface is fitted separately"[167], resulting in an hyperbola that gives the "correct asymptotic representation of the variance when log-strike tends to plus or minus infinity"[168]

2: Constraints are imposed that guarantee no vertical and no horizontal arbitrage opportunities[169]

The following equation results from the stochastic volatility inspired parametrisation:

$$\begin{aligned}\sigma^2[x] &\triangleq v(\{m,s,a,b,\rho\},x)\\ &= a + b\left(\rho(x-m) + \sqrt{(kx-m)^2+s^2}\right)\end{aligned} \quad (129)$$

"where a, b, ρ, s are parameters which are dependent on the time slice and x = ln (K/F)"[170].

One problem with such parametrisation is that it might present arbitrage situations at times. However, advantages include: quick computations, good approximations for implied volatilities for deep in-the-money and deep out-of-the-money options. And the stochastic volatility inspired fit for the equity markets is better than that for energy markets, "for which Deryabin[171] reported an error of maximum 4-5% for front year and respectively 1-2% for long maturities"[172].

The original quasi explicit calibration procedure is based on "matching input data $\{\sigma_i^{MKT}\}_{i=1...M'}$"[173], and becomes an optimisation problem:

$$\begin{aligned}\min_{\{a,b,\rho,m,s\}} \sum_{i=1}^{N} \Bigg(v\,\Bigg[\{m,s,a,b,\rho\},\\ ln\left(\frac{K_i}{F}\right)\Bigg] - (\sigma_i^{MKT})^2\Bigg)^2\end{aligned} \quad (130)$$

If we set the focus on total variance V = vT, the stochastic volatility inspired model transforms to the following equation:

$$V(y) = \alpha T + \delta y + \beta\sqrt{y^2-1} \quad (131)$$

where $y = \frac{x-m}{s}$, $\beta = bsT$, $\delta = \rho bsT$, $\alpha = aT$.

With the notation $\bar{V}_i = [\sigma_i^{MKT}]^2 T$, "for a given T, and s, which is transformed into $\{y_i, \hat{V}_i\}$, we look for the solution of the 3-dimensional problem"[174]:

$$\begin{aligned}&\min_{\{\beta,\delta,\alpha\}} F_{\{y_i,v_i\}}(\beta,\delta,\alpha)\\ &\quad= \sum_{i=1}^{N} w_i\left(\alpha + \delta y_i + \beta\sqrt{y_i^2+1} - \hat{V}_i\right)^2\end{aligned} \quad (132)$$

[161] B. Dumas, J. Fleming and R.E. Whaley - Implied volatility functions: Empirical tests. *The Journal of Finance. (1998)*
[162] C. Humescu - *Implied volatility surface: construction methodologies and characteristics. (2011)*
[163] Ibidem
[164] S. Borovkova and F. J. Permana - *Implied volatility in oil markets. Computational Statistics and Data Analysis. (2009)*
[165] C. Humescu - *Implied volatility surface: construction methodologies and characteristics. (2011)*
[166] Ibidem
[167] Ibidem
[168] Ibidem
[169] Ibidem
[170] Ibidem
[171] M.V. Deryabin - *Implied volatility surface reconstruction for energy markets: spot price modeling versus surface parametrisation. (2011)*
[172] C. Humescu - *Implied volatility surface: construction methodologies and characteristics. (2011)*
[173] Ibidem
[174] Ibidem

with the following domain being valid:

$$\begin{cases} \beta_{MIN} \leq \beta \leq 4s \\ -\beta \leq \delta \leq \beta \\ -(4s-\beta) \leq \delta \leq (4s-\beta) \\ \alpha_{MIN} \leq \alpha \leq \hat{V}_{MAX} \end{cases}$$

Given the solution $\{\beta^*, \delta^*, \alpha*\}$, we can find the corresponding values $\{a^*, b^* \rho^*\}$ and solve directly the 2-dimensional optimisation problem:

$$\min_{\{m,s\}} \sum_{i=1}^{N} \left(v\left[\{m,s,a^*,b^*,\rho^*\}, ln\left(\frac{K_i}{F}\right)\right] - v_i^{MKT}\right)^2 \quad (133)$$

The idea behind this 2-parameter optimisation added to the first 3-parameter optimisation is that the "procedure is much less sensitive to the choice of initial guess, and the resulting parameter is more reliable and stable"[175]. Knowing that the SVI is performed "sequentially, expiry by expiry"[176], enhanced procedures exist which satisfy the no-calendar arbitrage and the no-strike arbitrage simultaneously.

7.4.3 Entropy-based parametrisation

Entropic calibrations, used for thorough risk-neutral price distributions, implied volatility functions and option pricing functions is "an algorithm that yields arbitrage-free diffusion process by minimising the relative entropy distance to a prior diffusion"[177], and it is used to interpolate between implied volatilities of the options that are being traded.

Entropy maximisation (which aims to build the risk-neutral implied probability density function for the end price of the asset) will not present over fitting problems, is flexible and "can be applied to a wider range of calibration situations"[178].

Most entropy calibrations in financial modeling "use the logarithmic measure of Shannon and Wiener"[179]. A problem with such types of paremetrizations is that if it only uses prices of vanilla options to apply the entropy maximisation, then the density function will be exponential. However, they do respect the power laws of the Zipf-Mandelbrot type. An example of such entropies that respects the Zipf-Mandelbrot laws is Renyi entropy[180], which is used to get interpolations in an arbitrage-free environment. The main idea behind the interpolations is that "there is a one-to-one correspondence between the pricing formula for vanilla options and the associated gamma"[181]. Therefore, if we have the option gamma we can come back to the equivalent option pricing equation.

Given the following strikes, $K_j, j = 1, ..., M$, the density function of the Renyi entropy is the following:

$$p(x) = \left(\lambda + \beta_0 x + \sum_{j=1}^{M} \beta_j (x - K_j)^+\right)^{\frac{1}{\alpha-1}} \quad (134)$$

where $\alpha, \lambda, \beta_0, ..., \beta_M$ "are calibrated by matching the input prices computed using the previous density function"[182].

Humescu shows that for call options, we have to follow the following condition:

$$\bar{S}_0 - K_m - \frac{\alpha-1}{\alpha} \sum_{m=1}^{j-1} Y_m [X_m(x)]^{\frac{\alpha}{\alpha-1}} \left(x - K_m - \frac{\alpha-1}{2\alpha-1} Y_m X_m(x)\right) = |_{x=K_j}^{x=K_{j+1}} C_j^{MKT} \quad (135)$$

where

$$X_j(x) \triangleq \lambda + \sum_{j=1}^{M} \beta_j (x - K_j) \quad (136)$$

$$Y_j \triangleq \left(\sum_{m=0}^{j} \beta_m\right) \quad (137)$$

and imposing a normalisation condition:

$$\int_0^\infty p(x)dx = 1 \Longrightarrow \frac{\alpha-1}{\alpha} \sum_{j=0}^{M} Y_j [X_j(x)]^{\frac{\alpha}{\alpha-1}} |_{x=K_j}^{K_{j+1}} = 1 \quad (138)$$

The advantages of such method are various:

1: It is easy to use for calibration of binary options and variance swaps

2: "The procedure allows for accurate recovery of tail distribution of the underlying asset implied by the prices of the derivatives"[183].

A disadvantage is that the inputs must be coming from arbitrage-free situations, otherwise the algorithm will not work[184].

[175] Ibidem
[176] Ibidem
[177] Ibidem
[178] Ibidem
[179] Ibidem
[180] D.C. Brody, I.R.C. Buckley and I.C. Constantinou - *Option price calibration from Renyi entropy. (2007)*

[181] C. Humescu - *Implied volatility surface: construction methodologies and characteristics. (2011)*
[182] Ibidem
[183] Ibidem
[184] Ibidem

7.5 Volatility Surface based on nonparametric representations, smoothing and interpolation: arbitrage-free algorithms

Of essential importance is that the data is interpolated is arbitrage-free, otherwise the algorithms will not work at all. Kahale[185] "proposes an interpolation procedure based on piecewise convex polynomials, mimicking the Black-Scholes-Merton formula"[186], which results in an arbitrage-free call price and hence in an arbitrage-free volatility smile. Secondly, the total variance is interpolated throughout the strikes in a linear way. "Cubic B-splines interpolation was employed by Wang, Yin and Qi[187], with interpolation performed on option prices"[188].

On the other hand, Benko, Fengler, Hardle and Kopa[189] "suggest to directly smooth implied volatility parametrisation by means of constrained local quadratic polynomials"[190].

If we have M expiries $\{T_j\}$ and N strikes $\{x_i\}$ and if market data is $\{\sigma_i^{MKT}(T_j)\}$, then various approaches have been adopted, but for the purpose of this research paper, let us only consider the case where "each maturity is treated separately"[191]: for this case there has to be a minimisation of the following variables:

$$\min_{\{\alpha_0^j,\alpha_1^j,\alpha_2^{(j)}\}} \sum_{i=1}^{N} \left\{ \sigma_i^{MKT}(T_j) - \alpha_0^{(j)} - \alpha_1^{(j)} * (x_i - x) - \alpha_2^{(j)}(x_i - x)^2 \right\} \frac{1}{h} K\left[\frac{x_i - x}{h}\right] \quad (139)$$

where K is the kernel function. An example is the Epanechnikov model:

$$K(u) = 0.75(1 - u^2)\mathbf{1}[|u| \leq 1] \quad (140)$$

"with $\mathbf{1}$(A) denoting the indicator function for a set A and h is the bandwidth which governs the trade-off between bias and variance"[192].

There are other approaches[193] which are based on cubic splines smoothing of option prices. This implies (and is an advantage) that the input data must not necessarily have be free of arbitrage. "It employs cubic splines, with constraints specifically added to the minimisation problem in order to ensure that there is no arbitrage"[194]. A problem with these approaches is that the pricing function is not considered in the polynomials, and hence the call price is approximated.

Laurini [195] adopts "constrained smoothing B-splines. This approach permits to impose monotonicity and convexity in the smoothed curve. It allows to impose directly the shape restriction of no-arbitrage in the format of the curve"[196]. Problems with such models are that they need the "knots to be placed on a rectangular grid"[197].

Thin-spline representations of implied volatility surfaces have been accounted for as well by Brecher[198], where they have been used to get a pre-smoothed surface "that will be eventually used as a starting point for building a local volatility surface"[199].

Marunh[200] builds the volatility surface through generic volatility parametrisation for each expiry, "with no-arbitrage conditions in space and time being added as constraints, while a regularisation to the calibrating functional based on the difference between market implied volatilities and, respectively, volatilities given by parametrisation. [...] The resulting optimisation problem has a lot of sparsity/structure"[201] and this helps to obtain a great fit in an amount of time lower than a second.

7.6 The volatility surface based on a simpler numerical methods: the price-wise linear method

This method is relatively easy and straightforward to implement since it involves the following formula to find the option prices:

$$y = \frac{y_i(X_{i+1} - x) + y_{i+1}(x - x_i)}{x_{i+1} - x_i} \quad (141)$$

where x's are strike prices and y's are the option prices.

[185] N. Kahale - *An arbitrage-free interpolation of volatilities. (2004)*
[186] C. Humescu - *Implied volatility surface: construction methodologies and characteristics. (2011)*
[187] Y. Wang, H. Yin and L.Qi - *No-arbitrage interpolation of the option price function and its reformulation. (2004)*
[188] C. Humescu - *Implied volatility surface: construction methodologies and characteristics. (2011)*
[189] M. Benko, M.R. Fengler, W.K. Hardle, and M. Kopa - *On extracting information implied in options. (2007)*
[190] C. Humescu - *Implied volatility surface: construction methodologies and characteristics. (2011)*
[191] Ibidem
[192] Ibidem
[193] M.R. Fengler - *Option data and modeling BSM implied volatility. (2010)*
[194] C. Humescu - *Implied volatility surface: construction methodologies and characteristics. (2011)*
[195] M. Laurini - *Imposing no-arbitrage conditions in implied volatility surfaces using constrained smoothing splines. (2007)*
[196] C. Humescu - *Implied volatility surface: construction methodologies and characteristics. (2011)*
[197] Ibidem
[198] D. Brecher - *Pushing the limits of local volatility in option pricing. (2006)*
[199] C. Humescu - *Implied volatility surface: construction methodologies and characteristics. (2011)*
[200] J. Maruhn - *On-the-fly bid/ask-vol fitting with applications in model calibration. (2010)*
[201] C. Humescu - *Implied volatility surface: construction methodologies and characteristics. (2011)*

"It is also the only method that fits the original data exactly"[202]. Such model is used when the trader does not want to calibrate the model with respect to the market prices of swaptions with the prices that the trader already has. Indeed, not only "these prices are not always considered to be the best match of the market prices"[203], but also also "there might not exist prices for the strikes and maturities the trader is looking for"[204]. A problem with such interpolation method is, however, that it may produce arbitrage situations in the interpolated volatilities, "even if there is none in the original data"[205].

[202]D. Kohlberg - *The interest volatility surface. (2011)*
[203]Ibidem
[204]Ibidem
[205]Ibidem

8 Attempts at capturing the dynamics of the implied volatility surface - 1

Since "local variance is a conditional expectation of instantaneous variance, we can estimate local volatilities generated by a given stochastic volatility model; implied volatilities then follow. Given a stochastic volatility model, we can then approximate the shape of the implied volatility surface"[206].

8.1 Introduction to the first empirical application

The first empirical section of this research paper will analyse how the Heston, SABR and Monte Carlo model are used to price options and whether the models' prices are in line with the market prices. For the purpose of this research paper, it must be underlined that the following factors have been used in order to calculate the Heston, SABR and Monte Carlo implied volatilities on the software MATLAB 2016[207]:

1: Implied volatilities, strike prices and maturities gotten form Yahoo! Finance in the "Option Chain" section of the website of various stocks;

2: Data obtained exclusively for high volumes of open interest exists for the options being traded in the market;

3: Constant strikes throughout the evolution of the spot price, so as to always compare the same options implied volatilities surfaces and prices;

4: Average of the implied volatilities of the two previous underlying moves in the same direction and with similar magnitude; such implied volatilities have been used to calibrate the Heston and SABR model as initial volatilities. The idea is to transform these average implied volatilities in order to have new implied volatilities which, thanks to the Heston and SABR model, take into account the fact the volatility would be stochastic;

5: Interpolation to find missing volatilities for options (for given maturities and strikes) by using the sum of differences for volatilities at the extremes of a curve for the same maturity or an average for volatilities in between strike prices;

6: For the Monte Carlo implied volatilities found, if abnormal values are a result of the calculations, then the average of the closest implied volatilities for the same strike are calculated. If the abnormal value is found at one of the extremes of points of the surface (for example for the first time to maturity implied volatility with strike of $X), then the difference between the two next closest implied volatilities with the same strike is taken and added to the implied volatility value calculated for the option with the closest time to maturity (given the same strike $X);

7: Risk-free rate of 0.25% (which is the current yield of 10-year German Bunds);

8: Dividend yields according to what is found on the respective Yahoo! Finance pages of each company;

9: T (time to maturity of the options) between 1 month and 2 years;

10: 10000 iterations to calculate the Heston implied volatility;

11: For the Heston model, v = 0.5 kappa = 0.5, theta = 0.5, volatility of volatility = 0.05, rho = 0.5, with the respective lower and upper bounds of [0,1], [0,100], [0,1], [0,0.5], [-0.9,0.9];

12: 100000 simulations for the Monte Carlo process;

13: Assumption that the underlying has moved without jumps throughout the specific trading day, since we are testing models which do not account for jumps.

The MATLAB 2016 codes used in order to calibrate the Heston and SABR models and the VBA code used for the Monte Carlo process are respectively in Appendix A, Appendix B and Appendix C:

The way the Heston, SABR and Monte Carlo will be compared to the market implied volatilities and prices is through analysing how they price options and estimate implied volatilities given various swings in the market prices of the underlying of the specific options. The "swings" in the daily underlying price movement will be divided in the following categories:

A: -5.00% to -1.00%;

B: -1.00% to 0.00%;

C: 0.00% to +1.00%;

D: +1.00% to +5.00%;

We do not include daily absolute changes in percentage of the underlying price of more than 5% so as to exclude jumps, since the Heston and SABR model do not account for jumps. Moreover, there has been the decision to differentiate between a underlying movement of less than 1% and of more than 1% because in "normal times" the stock price will move by less than 1%, while in specific times or for specific catalysts the underlying price might show more nervous movement, depreciating or appreciating by more than 1%.

Lastly, it must be stated that the comparison of the Heston implied volatility and the actual (Black Scholes) implied volatility retrieved by Yahoo! Finance

[206]J. Gatheral - *The volatility surface: a practitioner's guide. (2006)*

[207]Heston and SABR MATLAB 2016 codes obtained respectively at the
following websites:
1] http://it.mathworks.com/matlabcentral/fileexchange/29446-heston-model-calibration-and-simulation,
2] http://fr.mathworks.com/help/fininst/calibrating-the-sabr-model.html?refresh=true
Monte Carlo and Black-Scholes VBAs obtained by HEC Paris Professor Olivier Bossard

is carried out on a benchmark stock, which in this research paper it will be Citigroup. The test will be also carried out on "comparable companies" of Citigroup (it will be explained in detail afterwards why each company is a comparable of Citigroup): Goldman Sachs, Zions Bancorporation, Google and Exxon Mobil.

8.2 Citigroup

Citigroup is the benchmark for our study. It is a large market capitalisation US bank ($115 billion[208]) with headquarters in New York and employs 240000 people[209].

It has been chosen as a validate candidate for this study because there is a lot of data available on the Internet concerning its option prices and implied volatilities (all of the following results are based on strikes, maturities, market prices and implied volatilities found at the following footnote[210]) and because there is sufficient data also for valid comparable companies to draw more thorough conclusions about the differences between the Heston, SABR and Monte Carlo models.

A dividend of $0.20 per share[211] has been taken into account in the calculations of implied volatility surfaces and option prices.

8.2.1 Underlying depreciation between 5% and 1%

In this study we will analyse the call and put option implied volatilities and prices given by the three models described previously. For both options, the average implied volatilities (shown below in Tables 2 and 4) and the consequent SABR parameters (Tables 3 and 5) have been calculated by averaging the implied volatilities of closing of business days March 23^{rd} 2016 and April 5^{th} 2016, where the underlying has respectively depreciated by 2.33% and 1.31%. The realised implied volatilities and prices of the option will be for the closing of business April 7^{th} 2016, where the stock price decreased by 3.80%.

Let us analyse the performance of the three models firstly for the call:

Table 2: Citigroup Mean of Implied Volatilities (%) - Call

	May/16	Sep/16	Jan/17	Jan/18
35	44.34	44.80	36.17	36.34
40	31.79	30.07	30.32	33.61
45	27.15	26.42	27.33	29.60
50	27.49	23.95	25.08	26.86
55	33.60	22.71	23.20	25.75

[208] http://finance.yahoo.com/q?s=C
[209] http://www.forbes.com/companies/citigroup/
[210] http://finance.yahoo.com/q/op?s=C+Options
[211] http://finance.yahoo.com/q?s=C

Table 3: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
May/16	0.20	0.5	-0.59	2.41
Sep/16	0.21	0.5	-0.76	1.74
Jan/17	0.19	0.5	-0.64	0.84
Jan/18	0.21	0.5	-0.65	0.61

Figure 9: Citigroup Market Implied Volatility (%/100) - Call - 07/04/16

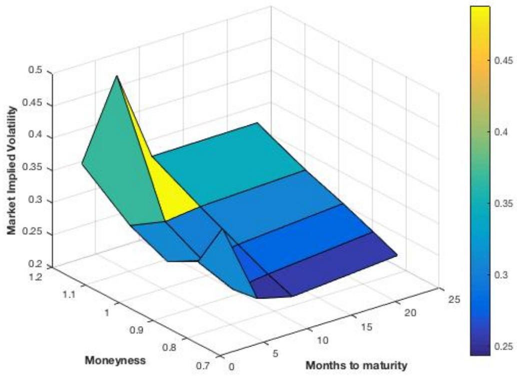


Figure 10: Citigroup Heston Implied Volatility (%/100) - Call - 07/04/16

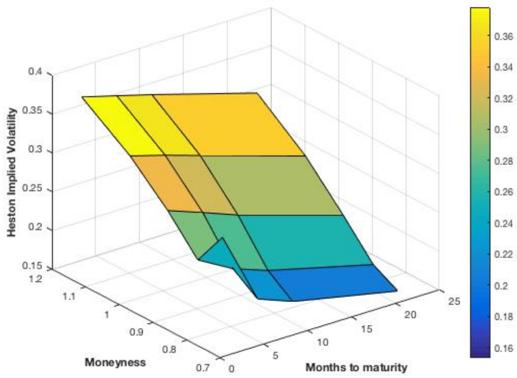


Figure 11: Citigroup SABR Implied Volatility (%/100) - Call - 07/04/16

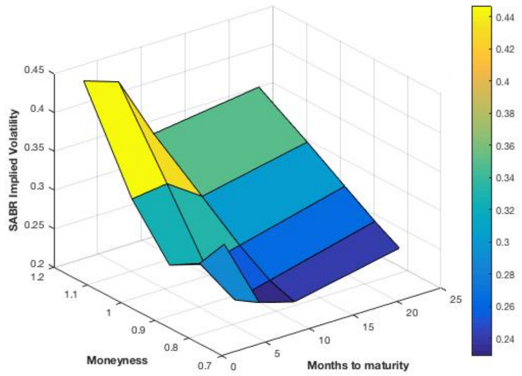
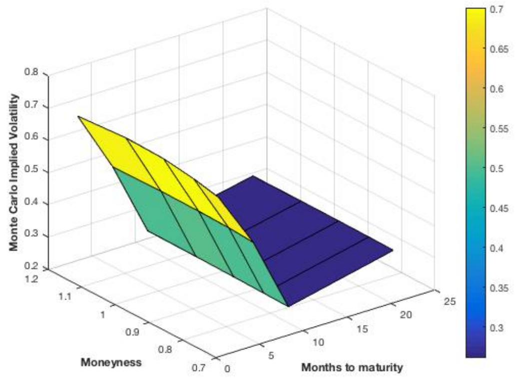


Figure 12: Citigroup Monte Carlo Implied Volatility (%/100) - Call - 07/04/16



By comparing Figures 9, 10, 11 and 12, we can spot the fact that the SABR model almost perfectly matches the implied volatilities shown by the market for all strikes and maturities. It overstates implied volatility when the call is far in-the-money with one month to maturity, but for the rest of the performance it manages to portray a skew for the call having 2 years to maturity. Moreover, the smile for the call having one month to maturity is almost perfectly shown.

Similar results can be said for the Heston model, which however fails to capture the spike in volatility for the call having a few months to maturity and being in-the-money and understated implied volatility for all strikes and maturities.

Lastly, the Monte Carlo model overestimates implied volatility for the call having one month to maturity and does the opposite when the option has 2 years to maturity. It fails to show the smile for the very short maturity and does not show either the skew for the call having 2 years to maturity.

Therefore, the clear winner in terms of approaching the market volatilities shown by the market in this case is the SABR model.

Figure 13: Citigroup Heston - Market price differential ($) - Call - 07/04/16

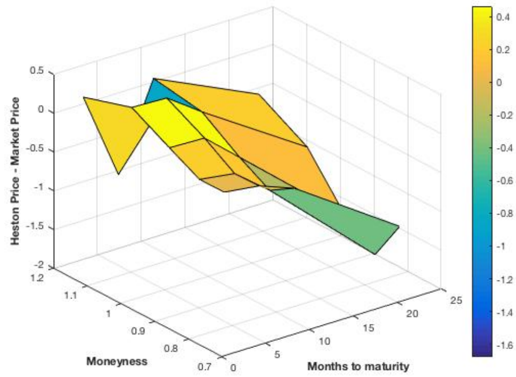


Figure 14: Citigroup SABR - Market price differential ($) - Call - 07/04/16

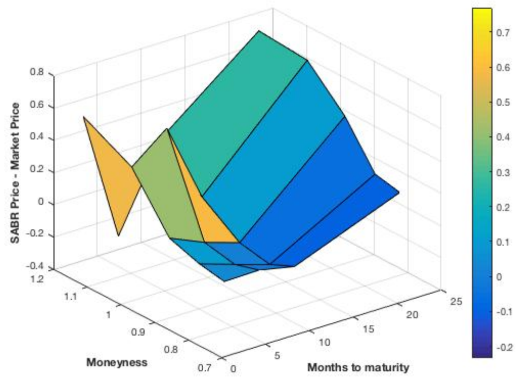
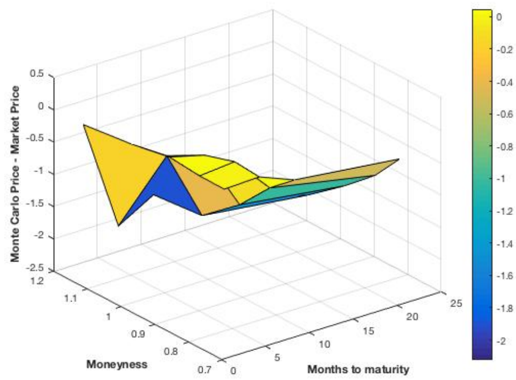


Figure 15: Citigroup Monte Carlo - Market price differential ($) - Call - 07/04/16



When comparing the performances of the three models in terms of pricing the options (Figures 13 to 15), the SABR model is again the clear winner. Indeed, the price differentials with respect to the observed market prices are the smallest of the three models, reaching up to an overestimation of $0.6 per option for very long maturities. The Heston and Monte Carlo models, on the other hand, although they perform well at pricing calls close to maturity, they both underprice the options, arriving to price differentials of -$1 and -$2 respectively for longer maturities.

Let us now compare the three models with respect to the put:

Table 4: Citigroup Mean of Implied Volatilities (%) - Put

	May/16	Sep/16	Jan/17	Jan/18
20	69.93	56.45	51.95	44.27
25	58.40	48.29	43.90	40.30
30	47.17	38.89	38.13	35.07
35	38.06	33.75	33.16	31.05
40	31.28	29.30	29.57	30.45
45	27.67	23.95	26.52	25.48

Table 5: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
May/16	0.19	0.5	-0.46	1.58
Sep/16	0.18	0.5	-0.64	0.97
Jan/17	0.18	0.5	-0.45	0.82
Jan/18	0.19	0.5	-0.99	0.35

Figure 16: Citigroup Market Implied Volatility (%/100) - Put - 07/04/16

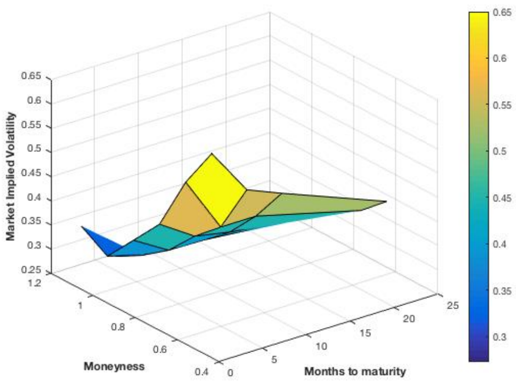


Figure 17: Citigroup Heston Implied Volatility (%/100) - Put - 07/04/16

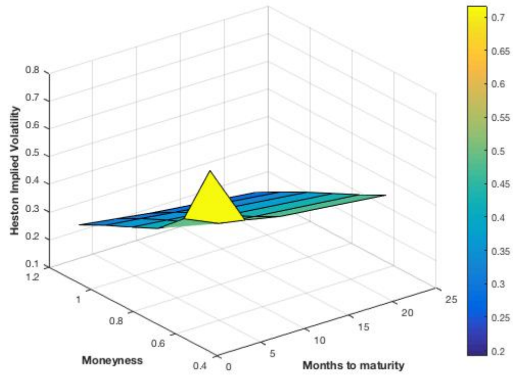


Figure 18: Citigroup SABR Implied Volatility (%/100) - Put - 07/04/16

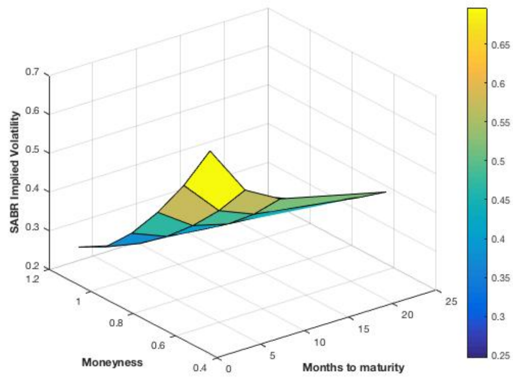
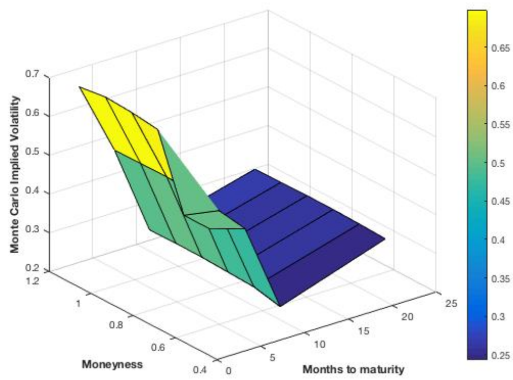


Figure 19: Citigroup Monte Carlo Implied Volatility (%/100) - Put - 07/04/16



On the other hand, when analysing the put (Figures 16 to 19), the SABR model is not a clear winner in terms of estimating implied volatility. Indeed the

Heston model is quite precise too: both this model and the SABR one manage to capture the skew for the put having very long maturities and the spike in implied volatility when the option is very out-of-the-money with one month to maturity. However they both fail to capture the smile for the put having one month to maturity.

The Monte Carlo model does a worse job in finding implied volatility: it indeed overestimates it for the put having one month to maturity, underestimates it for the options with one year to maturity and approximates it for the option having longer maturities (where however it fails to capture the smile seen in the market implied volatilities).

Figure 20: Citigroup Heston - Market price differential ($) - Put - 07/04/16

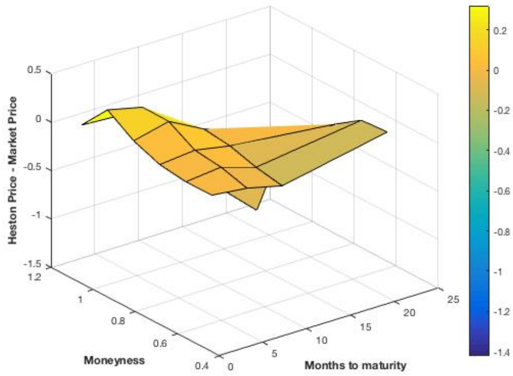


Figure 21: Citigroup SABR - Market price differential ($) - Put - 07/04/16

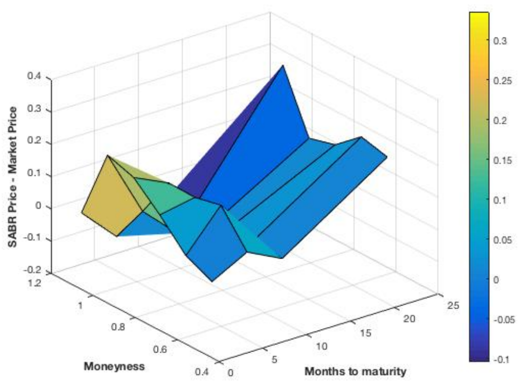
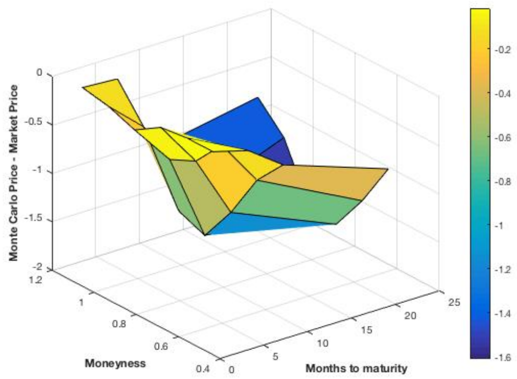


Figure 22: Citigroup Monte Carlo - Market price differential ($) - Put - 07/04/16



When comparing the pricing performance of the three models (Figures 20 to 22), we can see that the Heston model does a great job at all strikes and maturities except when the option is far in-the-money with 2 years to maturity. Indeed, the differential there is of about an $1.4 underpricing by the model.

The SABR model is as precise as the Heston model, overpricing the put at an average of $0.1 for longer maturities and of less than $0.1 for shorter ones.

The Monte Carlo model on the other hand underprices the put greatly for all strikes and maturities (except when the option is far in-the-money with one month to maturity), up to an underpricing of $1.5.

Thus, the Heston model performs as greatly as the SABR model in this specific scenario.

8.2.2 Underlying depreciation between 1% and 0%

Here we will first see in Tables 6 to 9 the respective averaged implied volatilities and SABR calibrated parameters for call and put options after having retrieved that the underlying has depreciated by 0.74% on March 14^{th} 2016 and by 0.31% on March 31^{st} 2016. For the actual market data, the implied volatilities have been retrieved at the end of business day April 4^{th} 2016, where the underlying has depreciated by 0.97%.

Table 6: Citigroup Mean of Implied Volatilities (%) - Call

	May/16	Sep/16	Jan/17	Jan/18
35	43.97	37.32	34.85	35.36
40	33.30	30.87	31.04	32.42
45	27.64	27.28	27.88	29.39
50	26.32	24.88	25.43	27.48
55	31.16	23.61	23.76	25.83

Table 7: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
May/16	0.19	0.5	-0.52	2.12
Sep/16	0.19	0.5	-0.59	1.02
Jan/17	0.19	0.5	-0.62	0.63
Jan/18	0.20	0.5	-0.59	0.52

Figure 23: Citigroup Market Implied Volatility (%/100) - Call - 04/04/16

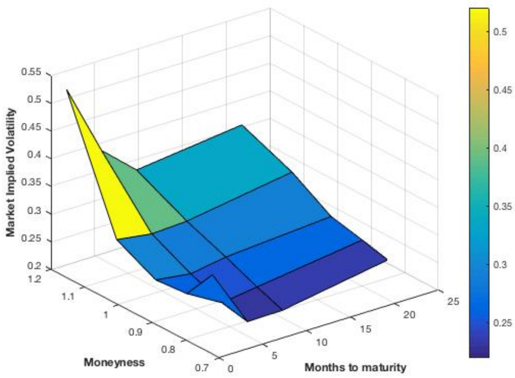


Figure 24: Citigroup Heston Implied Volatility (%/100) - Call - 04/04/16

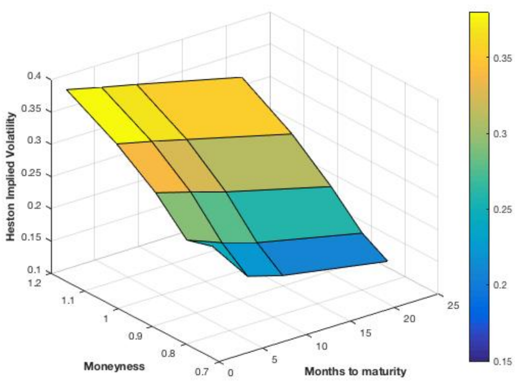


Figure 25: Citigroup SABR Implied Volatility (%/100) - Call - 04/04/16

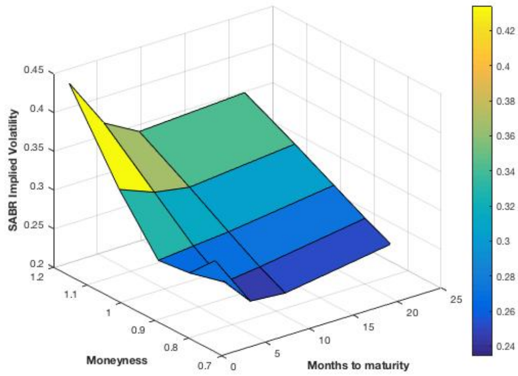
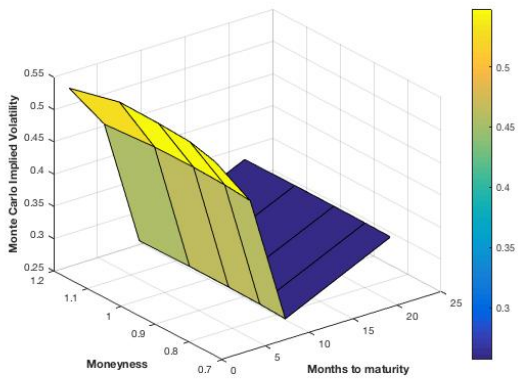


Figure 26: Citigroup Monte Carlo Implied Volatility (%/100) - Call - 04/04/16



Once again, we can see that the SABR model (Figure 25) estimates implied volatility identically to the implied volatilities found in the market (Figure 23). It portrays the smile for the call having one month to maturity and the skew for the call having 1 year and 2 years to maturity. Moreover, the values of implied volatilities are neither underestimated nor overestimated by the model.

The Heston model (Figure 24), on the other hand, while it correctly shows the smile and skews for the respective maturities, underestimates the implied volatility for the call being far in-the-money with one month to maturity. That is the only mistake committed by the model.

The Monte Carlo model (Figure 26) overestimates volatility for very short maturities and does the opposite when the call has one year to maturity. Therefore, it does a poor job in approximating implied volatility here too.

Therefore, although not by far, the SABR model is the winner in this case.

Figure 27: Citigroup Heston - Market price differential ($) - Call - 04/04/16

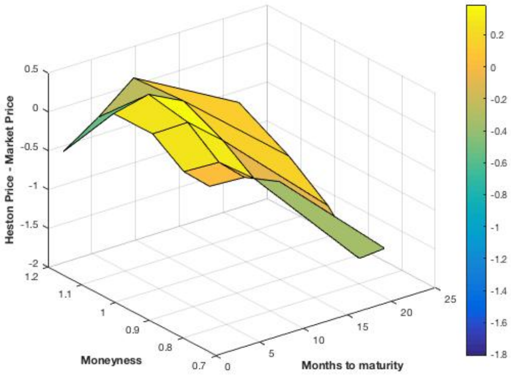


Figure 28: Citigroup SABR - Market price differential ($) - Call - 04/04/16

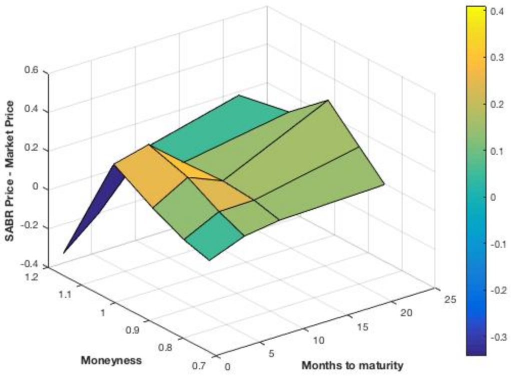
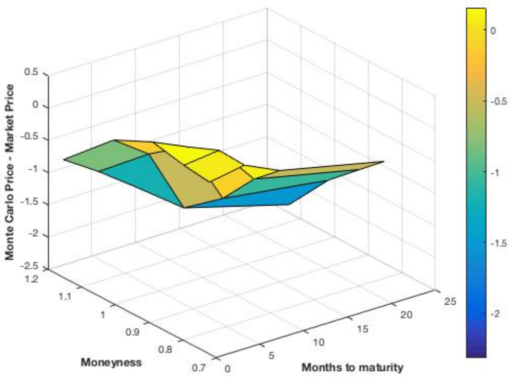


Figure 29: Citigroup Monte Carlo - Market price differential ($) - Call - 04/04/16



When comparing the pricing performances of the three models (Figures 27 to 29), we can see that the SABR model matches the prices the closest to those of the market. Indeed, it overprices the call at all strikes and maturities by $0.1, while it underprices it when it has one month to maturity and is very in-the-money by $0.3.

The Heston and Monte Carlo models, on the other hand, arrive to underestimate the call by about $1.5 for longer maturities and by about $0.1 and $0.5 respectively when the call has one month to maturity. Therefore, the Heston model dominates the Monte Carlo one here.

Let us see how the three models compare when analysing the put implied volatilities and prices:

Table 8: Citigroup Mean of Implied Volatilities (%) - Put

	May/16	Sep/16	Jan/17	Jan/18
20	69.34	58.06	52.10	45.62
25	61.14	50.05	44.77	42.21
30	47.85	40.69	38.64	35.76
35	38.33	34.74	33.81	32.22
40	31.78	30.31	30.11	29.60
45	28.23	26.83	27.03	27.71

Table 9: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
May/16	0.19	0.5	-0.38	1.62
Sep/16	0.18	0.5	-0.36	1.13
Jan/17	0.18	0.5	-0.44	0.80
Jan/18	0.18	0.5	-0.41	0.57

Figure 30: Citigroup Market Implied Volatility (%/100) - Put - 04/04/16

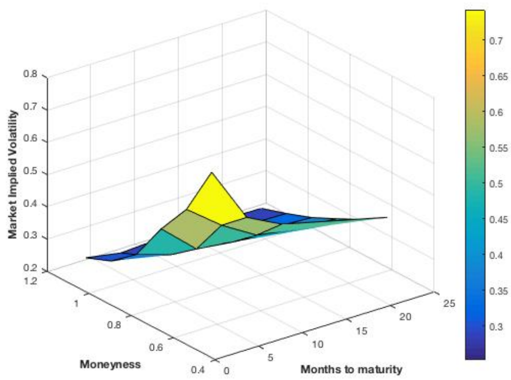


Figure 31: Citigroup Heston Implied Volatility (%/100) - Put - 04/04/16

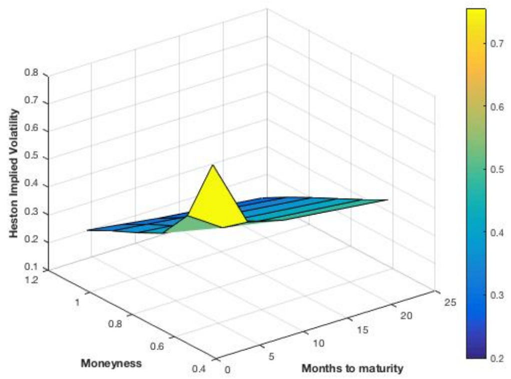


Figure 32: Citigroup SABR Implied Volatility (%/100) - Put - 04/04/16

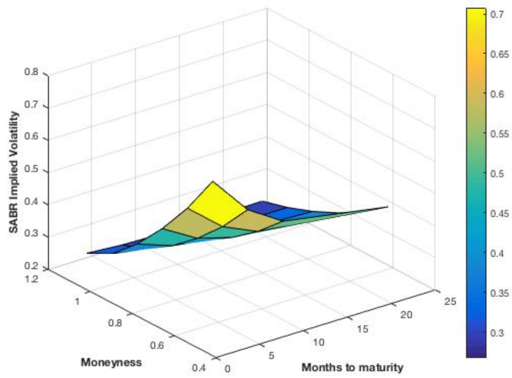
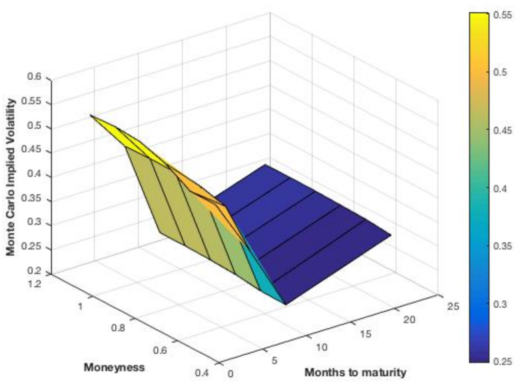


Figure 33: Citigroup Monte Carlo Implied Volatility (%/100) - Put - 04/04/16



Once again, as for the case of the underlying depreciating between 5% and 1%, here as well both the Heston and SABR models (Figures 31 and 32 respectively) are practically perfect at estimating the implied volatilities at all strikes and maturities. They capture the big spike in implied volatility for the put being very out-of-the-money with one month to maturity and they manage to show the skews for all maturities.

The Monte Carlo model (Figure 33) instead overestimates implied volatility for very short times to maturity and does a better job for longer maturities. However, its performance is far worse than those of the Heston and SABR models.

Figure 34: Citigroup Heston - Market price differential ($) - Put - 04/04/16

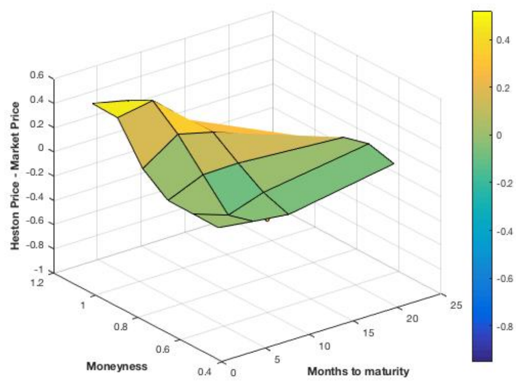


Figure 35: Citigroup SABR - Market price differential ($) - Put - 04/04/16

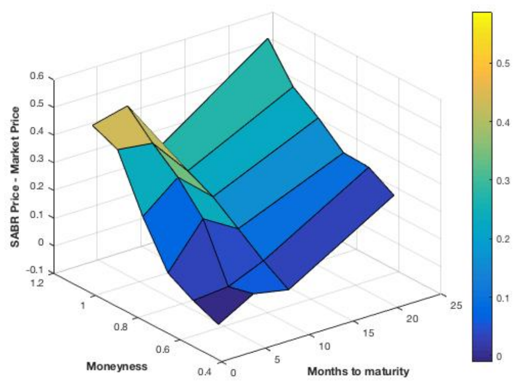
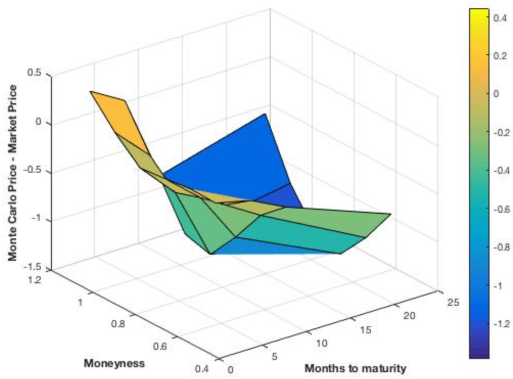


Figure 36: Citigroup Monte Carlo - Market price differential ($) - Put - 04/04/16



Surprisingly enough, although the Heston model and the SABR model perform equally when estimating the implied volatility of the put, when comparing them in terms of pricing (Figures 34 and 35), the Heston model does a better job. Indeed, it prices the put correctly for all maturities and strikes, except when the option is far in-the-money with two years to maturity (with a respective underpricing of $0.5).

The SABR model, on the other hand, overestimates the put price at almost all strikes and maturities by a greater and greater amount as the option becomes more and more in-the-money (arriving to an overpricing of $0.5 for longer maturities).

Lastly, while the Monte Carlo model is relatively precise for short maturities, it greatly underprices the put for longer ones, especially when the option is at-the-money.

8.2.3 Underlying appreciation between 0% and 1%

For this study we will analyse the Heston, SABR and Monte Carlo performances for an underlying appreciation of 0.84% occurred on closing of business day April 6^{th} 2016.

Before going into the detail, let us state that for the Heston and SABR model, the market implied volatilities for an underlying appreciation of 0.53% occurred on March 10^{th} 2016 and for an underlying appreciation of 0.38% occurred on March 30^{th} 2016 have been averaged to obtain specific surfaces for the call and put options (shown Tables 10 and 12) and SABR parameters (Tables 11 and 13).

Table 10: Citigroup Mean of Implied Volatilities (%) - Call

	May/16	Sep/16	Jan/17	Jan/18
35	43.42	37.38	35.67	35.03
40	32.41	31.20	31.50	32.79
45	28.10	27.67	28.55	29.68
50	27.15	25.40	26.11	27.70
55	31.06	24.15	24.29	26.50

Table 11: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
May/16	0.19	0.5	-0.48	2.06
Sep/16	0.19	0.5	-0.57	1.01
Jan/17	0.20	0.5	-0.61	0.65
Jan/18	0.20	0.5	-0.57	0.46

Figure 37: Citigroup Market Implied Volatility (%/100) - Call - 06/04/16

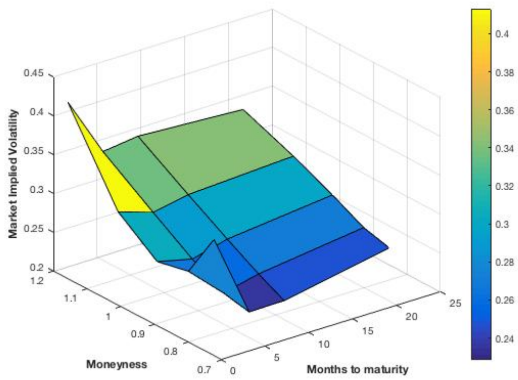


Figure 38: Citigroup Heston Implied Volatility (%/100) - Call - 06/04/16

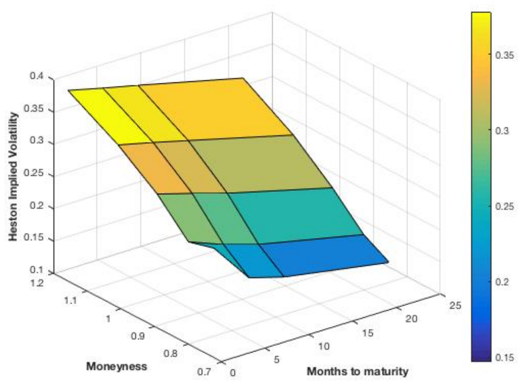


Figure 39: Citigroup SABR Implied Volatility (%/100) - Call - 06/04/16

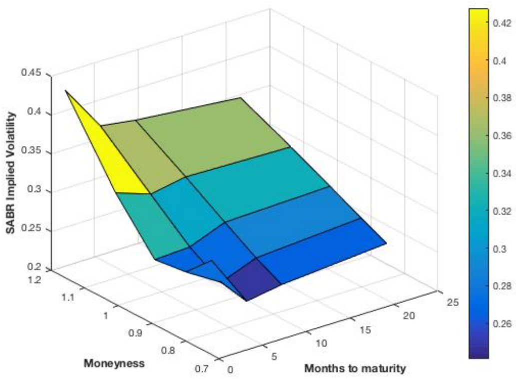
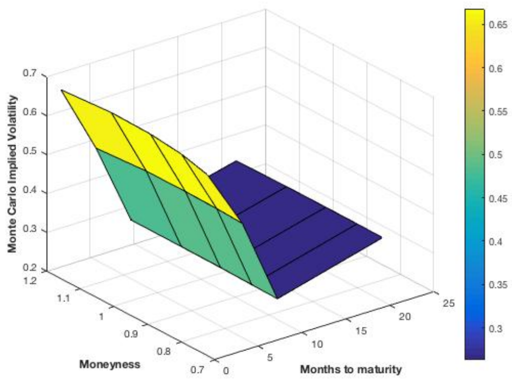


Figure 40: Citigroup Monte Carlo Implied Volatility (%/100) - Call - 06/04/16



We can see that the Heston and SABR models (Figures 38 and 39 respectively) do a great job at finding the implied volatilities for the call at all strikes and maturities. The only exception is for the Heston model, where the same happens as shown for the underlying depreciation between 1% and 0%: there is an underestimation of implied volatility for the call being in-the-money with one month to maturity.

The Monte Carlo model (Figure 40) fails to show the smile and skew respectively for one month to maturity and two years to maturity. Moreover, it overstates the implied volatility for very short maturities and is quite accurate for longer maturities. Therefore, also in this case there is an ample margin for improvement under the Monte Carlo model.

Figure 41: Citigroup Heston - Market price differential ($) - Call - 06/04/16

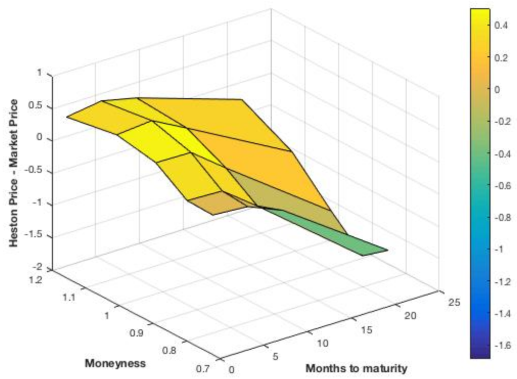


Figure 42: Citigroup SABR - Market price differential ($) - Call - 06/04/16

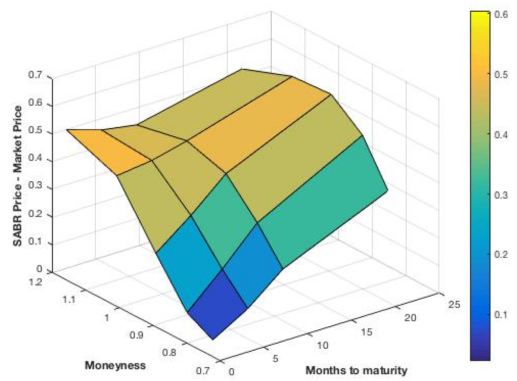
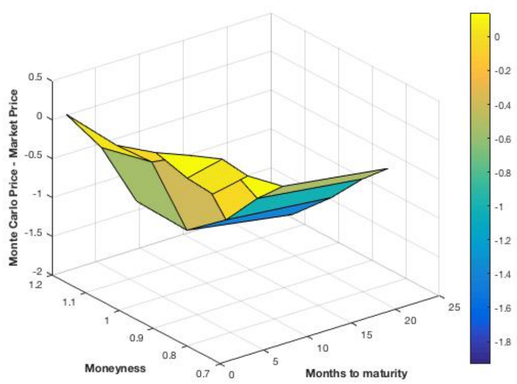


Figure 43: Citigroup Monte Carlo - Market price differential ($) - Call - 06/04/16



In terms of the pricing of the option (please refer to Figures 41 to 43), the SABR model is the clear winner: indeed, the absolute value in mispricing of the call does not pass the $0.5 figure, while it does go above $1 for both the Heston and Monte Carlo models, especially for long maturities. However, it must be stated that these last two models are quite precise for short maturities. They both underprice the call for long times to maturity, however.

Now, let us analyse the performances of the models with respect to the put:

Table 12: Citigroup Mean of Implied Volatilities (%) - Put

	May/16	Sep/16	Jan/17	Jan/18
20	68.26	57.86	52.47	46.06
25	59.97	50.23	45.16	42.32
30	47.95	40.95	39.18	36.94
35	38.44	35.11	34.30	32.80
40	32.04	30.82	30.67	30.15
45	28.44	27.74	27.91	28.60

Table 13: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
May/16	0.19	0.5	-0.41	1.52
Sep/16	0.18	0.5	-0.32	1.12
Jan/17	0.19	0.5	-0.39	0.81
Jan/18	0.19	0.5	-0.43	0.54

Figure 44: Citigroup Market Implied Volatility (%/100) - Put - 06/04/16

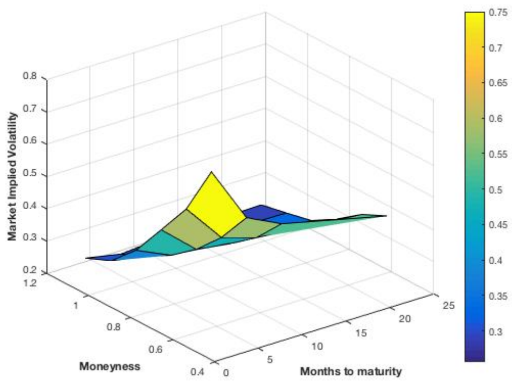


Figure 45: Citigroup Heston Implied Volatility (%/100) - Put - 06/04/16

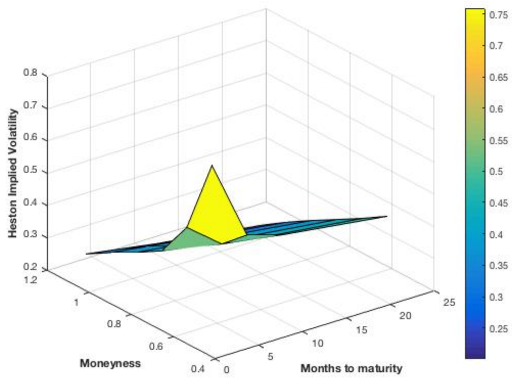


Figure 46: Citigroup SABR Implied Volatility (%/100) - Put - 06/04/16

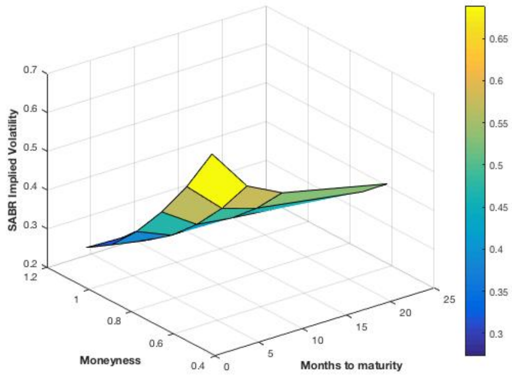
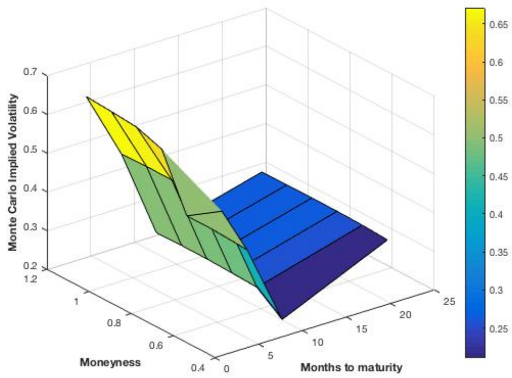


Figure 47: Citigroup Monte Carlo Implied Volatility (%/100) - Put - 06/04/16



Once again, the Heston model (Figure 45) performs equally compared to the SABR model (Figure 46) when finding the implied volatilities of the put. They estimate the implied volatilities correctly for all maturities and strikes, showing the skews at all times to maturity.

The Monte Carlo model (Figure 47), instead, overestimates implied volatility for very short maturities and correctly estimates it for longer ones.

Figure 48: Citigroup Heston - Market price differential ($) - Put - 06/04/16

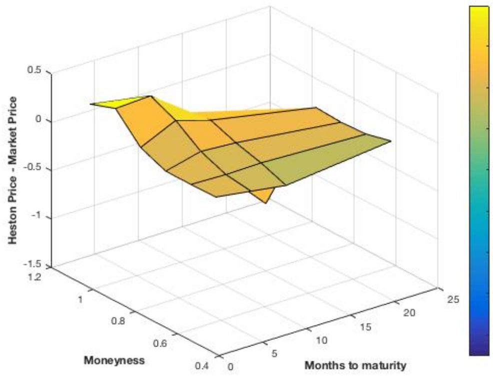


Figure 49: Citigroup SABR - Market price differential ($) - Put - 06/04/16

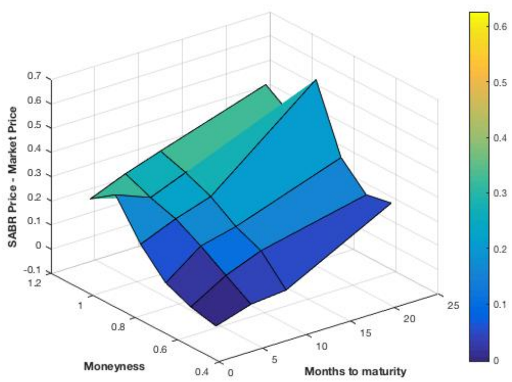
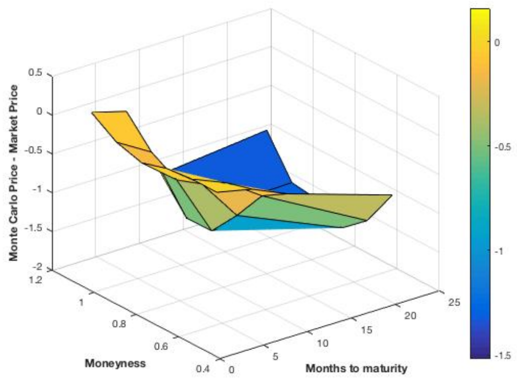


Figure 50: Citigroup Monte Carlo - Market price differential ($) - Put - 06/04/16



With respect to the pricing of the put, the Heston model (Figure 48) is the one that performs the best: indeed, it correctly prices the option at all maturities and strikes except when the option is very in-the-money with two years to maturity, in which case the mispricing is equivalent to an underestimation of $1.

SABR model-wise (Figure 49), while the mispricings never go above $0.7, they are almost everywhere, with the overestimations of prices increasing with both moneyness and time to maturity.

Lastly, the Monte Carlo model (Figure 50) does a great job at pricing short-dates puts, and underestimates puts with longer maturities, especially those that are near-the-money.

8.2.4 Underlying appreciation between 1% and 5%

For this study the average implied volatilities (Tables 14 and 16) and respective SABR parameters (Tables 15 and 17) have been calculated after a stock appreciation of 1.72% on April 1^{st} 2016 and of 1.61% on April 11^{th} 2016. The stock appreciation that has been recorded in order to retrieve the market implied volatility and price data occurred on April 14^{th} 2016, for an underlying appreciation of 1.65%.

Let us firstly see the performance of the three models for the call option:

Table 14: Citigroup Mean of Implied Volatilities (%) - Call

	May/16	Sep/16	Jan/17	Jan/18
35	47.71	38.25	34.31	33.19
40	30.89	29.67	30.07	31.88
45	26.71	25.97	27.13	27.97
50	27.54	23.28	24.69	27.27
55	33.99	22.61	22.99	25.32

Table 15: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
May/16	0.16	0.5	-0.17	2.89
Sep/16	0.17	0.5	-0.47	1.33
Jan/17	0.18	0.5	-0.53	0.71
Jan/18	0.19	0.5	-0.65	0.34

Figure 51: Citigroup Market Implied Volatility (%/100) - Call - 14/04/16

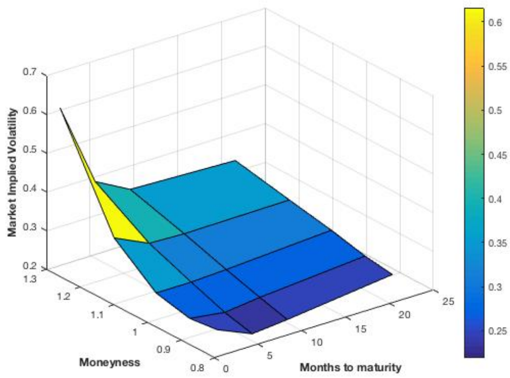


Figure 52: Citigroup Heston Implied Volatility (%/100) - Call - 14/04/16

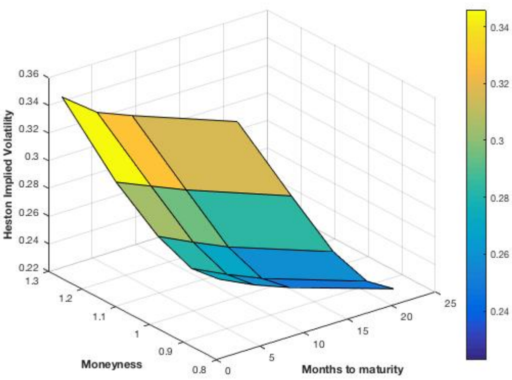


Figure 53: Citigroup SABR Implied Volatility (%/100) - Call - 14/04/16

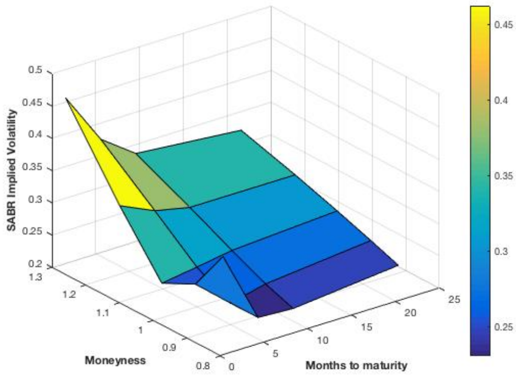
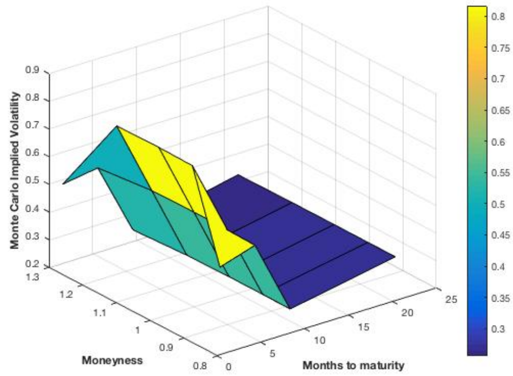


Figure 54: Citigroup Monte Carlo Implied Volatility (%/100) - Call - 14/04/16



In this case, unlike the other precedent ones, both the Heston (Figure 52) and SABR (Figure 53) models win the comparison. Indeed, they both fail to capture the spike in implied volatility when the option is far in-the-money with one month to maturity. However, they both estimate the skews correctly for all maturities.

Once again, the Monte Carlo model (Figure 54) over-estimates implied volatility for very short maturities and correctly measures it for longer ones, failing at the same time to show the skews for each time to maturity.

Figure 55: Citigroup Heston - Market price differential ($) - Call - 14/04/16

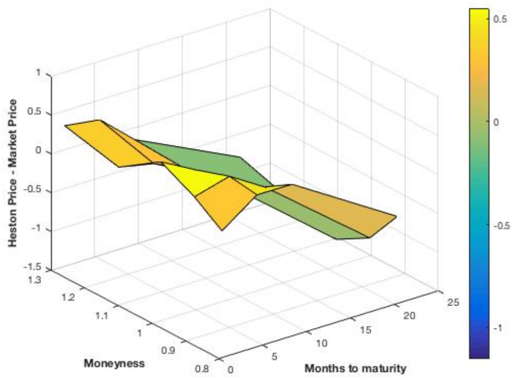


Figure 56: Citigroup SABR - Market price differential ($) - Call - 14/04/16

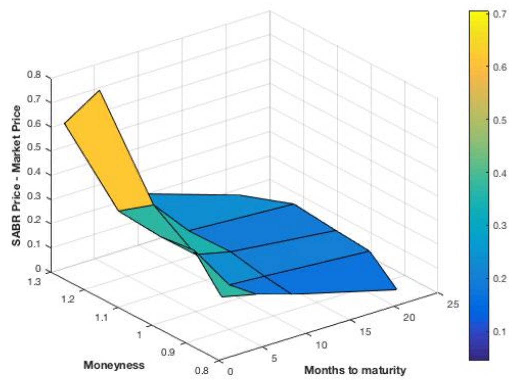
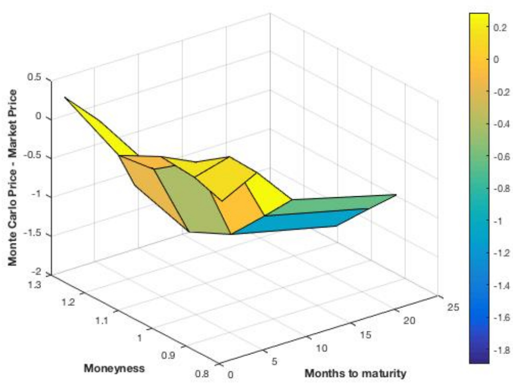


Figure 57: Citigroup Monte Carlo - Market price differential ($) - Call - 14/04/16



Even if the Heston model performs as well as the SABR model in terms of estimating implied volatilities for the call, it is worse than the Stochastic Alpha Beta Rho model when pricing the option. Indeed, both the Heston (Figure 55) and Monte Carlo (Figure 57) models are quite accurate for short maturities, with overestimations up to $0.3 for the Heston model and underestimations up to $0.3 for the Monte Carlo model. However, they both greatly underprice the call by over $0.5 for longer maturities, whereas the SABR model (Figure 56) overprices the call by $0.6 when it is far in-the-money with one month to maturity and by only $0.2 for the rest of strikes and maturities.

The following is the analysis of the performance of the three models with respect to the put option:

Table 16: Citigroup Mean of Implied Volatilities (%) - Put

	May/16	Sep/16	Jan/17	Jan/18
20	71.33	57.52	51.74	44.61
25	59.38	49.03	44.27	41.57
30	48.83	40.02	37.91	35.69
35	37.40	33.22	33.03	31.56
40	30.71	29.07	29.53	29.30
45	27.40	25.84	26.62	26.88

Table 17: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
May/16	0.18	0.5	0.04	2.06
Sep/16	0.16	0.5	-0.06	1.32
Jan/17	0.17	0.5	-0.20	0.92
Jan/18	0.18	0.5	-0.59	0.49

Figure 58: Citigroup Market Implied Volatility (%/100) - Put - 14/04/16

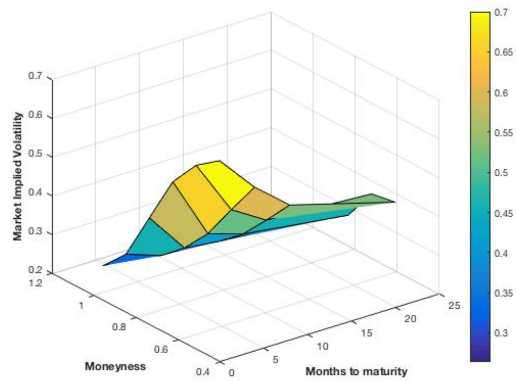


Figure 59: Citigroup Heston Implied Volatility (%/100) - Put - 14/04/16

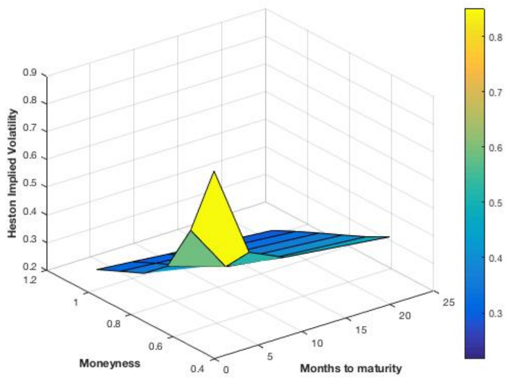


Figure 60: Citigroup SABR Implied Volatility (%/100) - Put - 14/04/16

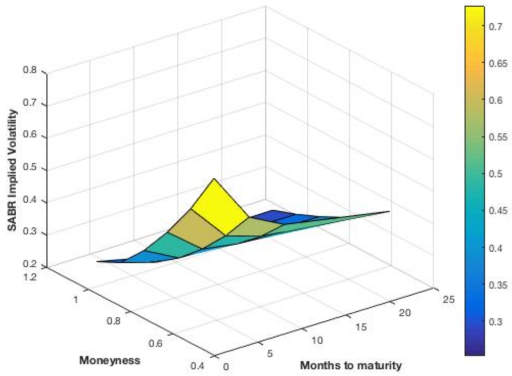
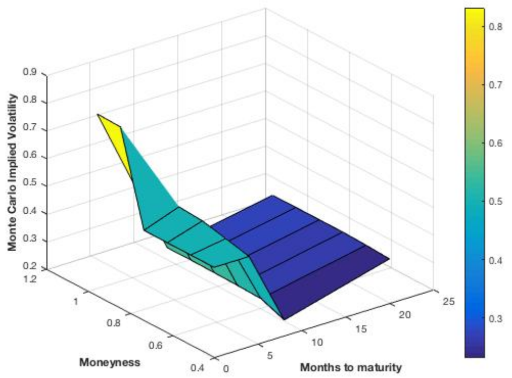


Figure 61: Citigroup Monte Carlo Implied Volatility (%/100) - Put - 14/04/16



We can see here that, although the Heston (Figure 59) and SABR (Figure 60) models perform equivalently in estimating implied volatilities, they are poorer at doing so than when the underlying appreciates by less than 1% or even depreciates. Indeed, the main difference compared to the other underlying movements lies in the underestimation of implied volatilities when the put has one month to maturity and is near-the-money.

Once again, the Monte Carlo methodology (Figure 61) does a poor job at estimating implied volatility, failing not only to capture the various skews according to times to maturity, but also by overestimating implied volatility greatly for short maturities.

Figure 62: Citigroup Heston - Market price differential ($) - Put - 14/04/16

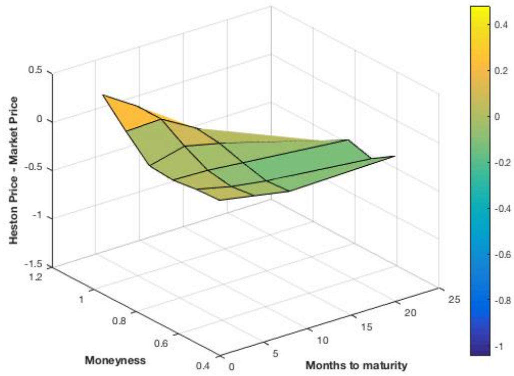


Figure 63: Citigroup SABR - Market price differential ($) - Put - 14/04/16

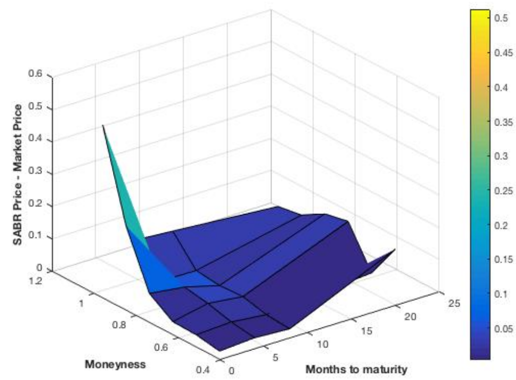
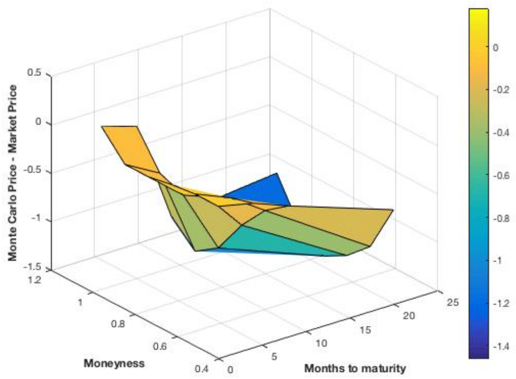


Figure 64: Citigroup Monte Carlo - Market price differential ($) - Put - 14/04/16



Surprisingly, the SABR model (Figure 63) is the winner when it comes to pricing the put option: it overprices it by less then $0.1 for all strikes and maturities, except for when the put is far in-the-money with one month to maturity, where the overpricing increases to $0.4.

The Heston model (Figure 62) is quite precise as well, but the absolute value in the price differential is more than twice that of the SABR model, reaching $0.2 in underestimation of the call price.

Lastly, the Monte Carlo model (Figure 64) is great at pricing very short-dated puts, but fails to price longer-dated puts (with the price differential with respect to market prices reaching almost $1.5 for near-the-money puts).

8.3 Goldman Sachs

The next study analyses the performance of the three models for Goldman Sachs, a large market capitalisation US bank ($69 billion[212]) with headquarters in New York, USA. It employs about 40000 people and is involved exclusively in investment banking activities, such as Merger and Acquisitions, Trading, Investing and Lending, and Investment Management[213].

It has been chosen as a validate comparable company because it is a very similar bank to Citigroup, with two exceptions: smaller market cap and no involvement in commercial banking.

Thus, the only ways that Goldman Sachs should be viewed differently from Citigroup is exclusively in the market cap and in the activities it is involved in. All of the following results are based on strikes, maturities implied volatilities and market prices found at the following footnote[214].

The dividend, found on Yahoo! Finance[215], that will be taken into account in the calculations of the volatility surfaces and prices is of $2.60 per share.

Let us now analyse the Heston, SABR and Monte Carlo performance for various underlying daily changes in value.

8.3.1 Underlying depreciation between 5% and 1%

For this section we will analyse the three models after an average of implied volatilities (Tables 18 and 20), with respective retrieval of SABR parameters (Tables 19 and 21 respectively for calls and puts), obtained by the stock depreciating by 1.53% on March 5^{th} 2016 and by 1.28% on April 4^{th} 2016. The market data has been obtained after the underlying has depreciated by 3.08% on April 7^{th} 2016.

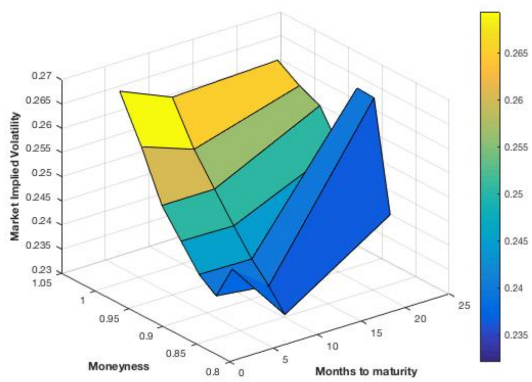
Table 18: Goldman Sachs Mean of Implied Volatilities (%) - Call

	Jul/16	Jan/17	Jan/18
150	25.70	24.95	28.43
155	24.40	24.30	24.48
160	23.36	23.68	25.64
165	22.55	23.09	26.63
170	21.87	22.61	25.95
175	21.41	22.17	23.62
180	21.23	21.77	25.04

Table 19: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
Jul/16	0.31	0.5	-0.57	1.15
Jan/17	0.30	0.5	-0.51	0.58
Jan/18	0.31	0.5	-0.36	0.94

Figure 65: Goldman Sachs Market Implied Volatility (%/100) - Call - 07/04/16



[212] http://finance.yahoo.com/q?s=GS
[213] http://finance.yahoo.com/q/pr?s=GS+Profile
[214] http://finance.yahoo.com/q/op?s=GS+Options
[215] http://finance.yahoo.com/q?s=GS

Figure 66: Goldman Sachs Heston Implied Volatility (%/100) - Call - 07/04/16

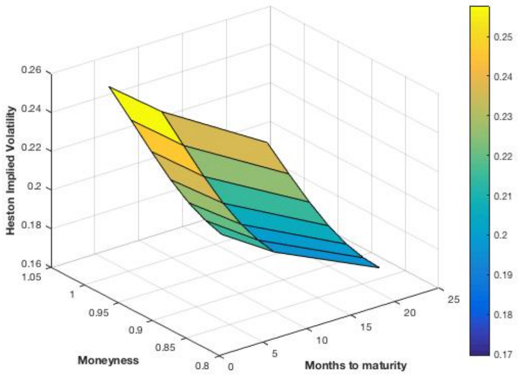


Figure 67: Goldman Sachs SABR Implied Volatility (%/100) - Call - 07/04/16

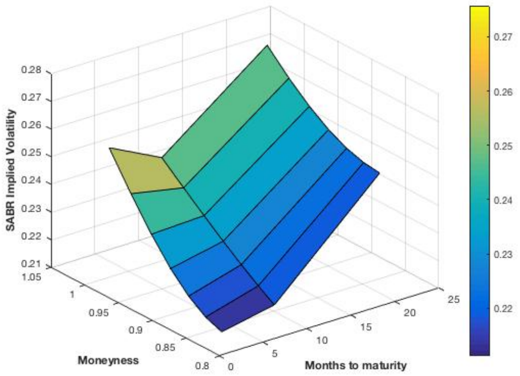
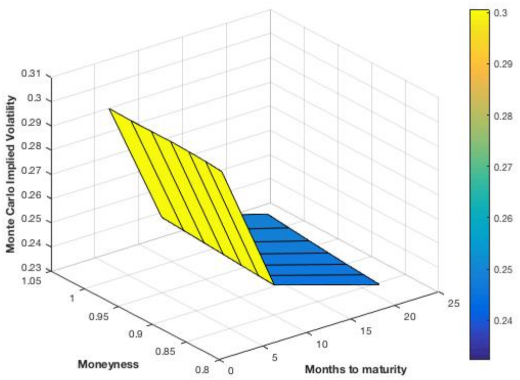


Figure 68: Goldman Sachs Monte Carlo Implied Volatility (%/100) - Call - 07/04/16



In terms of the call, we can see that the Heston model (Figure 66) slightly underestimates implied volatility at all strikes and maturities, However, it manages to show the skew for all maturities.

The same can be said for the SABR model (Figure 67), except for when it overestimates implied volatility when the call has two years to maturity.

The Monte Carlo model (Figure 68) on the other hand fails to show the skews at all maturities and overestimates volatility when the option has a few months to maturity.

Therefore, the Heston and SABR model are superior in this specific case.

Figure 69: Goldman Sachs Heston - Market price differential ($) - Call - 07/04/16

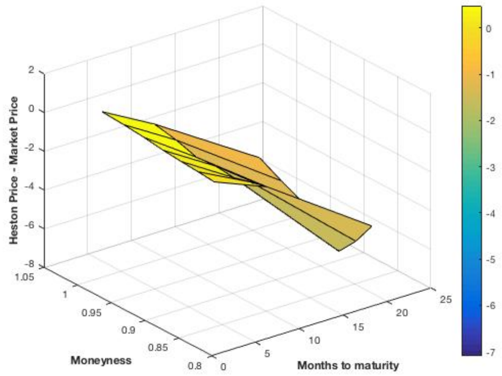


Figure 70: Goldman Sachs SABR - Market price differential ($) - Call - 07/04/16

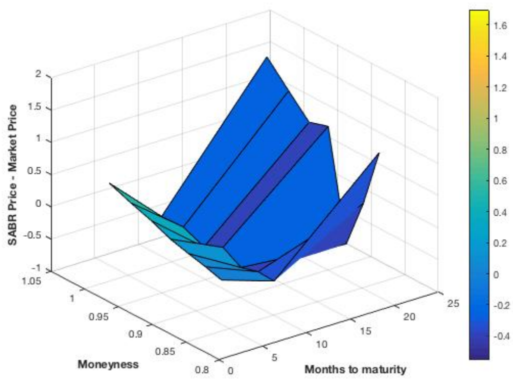
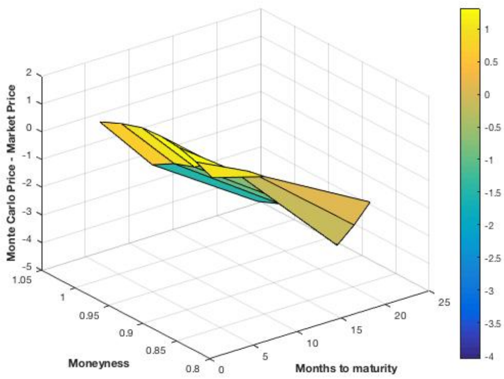


Figure 71: Goldman Sachs Monte Carlo - Market price differential ($) - Call - 07/04/16



In terms of differentials of prices, the Heston model (Figure 69) is quite precise when the option has a few months to maturity. On the other hand, as maturity increases, the differential becomes more and more negative, arriving to -$6.

The SABR model (Figure 70) instead is quite precise: indeed, for short maturities the absolute value in the differential does not go above $0.5, while for longer maturities it does not go above $1.5.

The performance of the Monte Carlo model (Figure 71) is slightly more precise than that of the Heston model, with underestimations of the call for longer maturities not going beyond the $5 figure.

Hence, the SABR model is superior in pricing the call for this particular case.

Table 20: Goldman Sachs Mean of Implied Volatilities (%) - Put

	Jul/16	Jan/17	Jan/18
130	32.43	30.19	29.19
135	30.63	29.17	28.42
140	29.09	28.26	28.38
145	27.60	27.35	26.50
150	26.40	26.68	30.92
155	25.21	25.88	26.69
160	24.24	25.25	25.05

Table 21: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
Jul/16	0.31	0.5	-0.49	1.21
Jan/17	0.32	0.5	-0.46	0.68
Jan/18	0.34	0.5	-0.94	0.14

Figure 72: Goldman Sachs Market Implied Volatility (%/100) - Put - 07/04/16

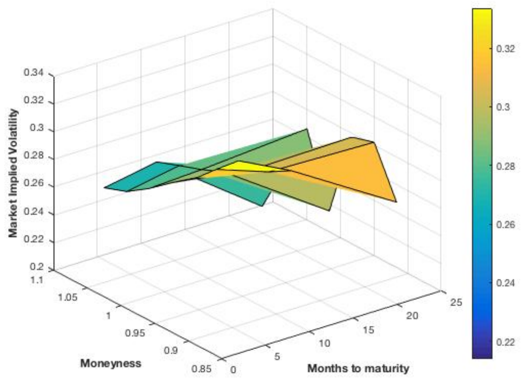


Figure 73: Goldman Sachs Heston Implied Volatility (%/100) - Put - 07/04/16

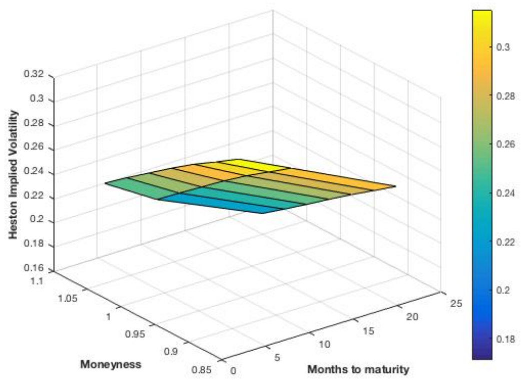


Figure 74: Goldman Sachs SABR Implied Volatility (%/100) - Put - 07/04/16

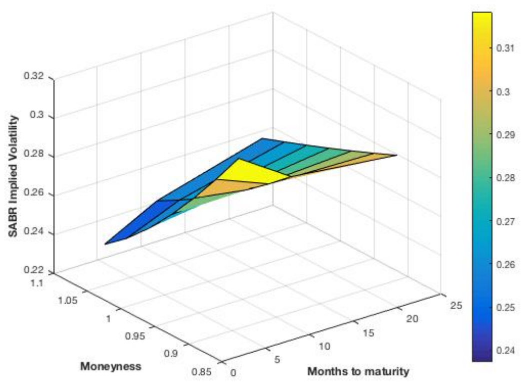
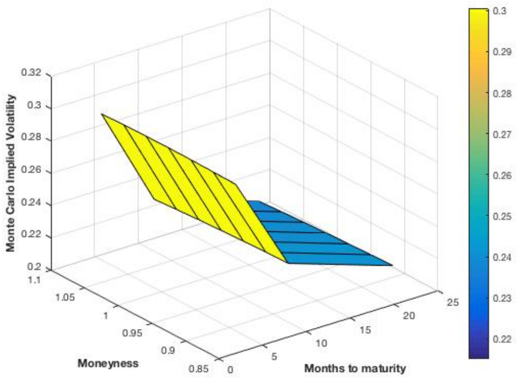


Figure 75: Goldman Sachs Monte Carlo Implied Volatility (%/100) - Put - 07/04/16



Also for the put option we can see that both the Heston (Figure 73) and SABR (Figure 74) models slightly underestimate volatility when compared to the market data, especially when the put is out-of-the-money. However, they both show the skews for shorter maturities correctly.

Monte Carlo-wise (Figure 75), implied volatility is overestimated for the put having a few months to maturity, and the opposite happens for longer maturities. Moreover, it fails to show the skew for the put having a few months to maturity.

Figure 76: Goldman Sachs Heston - Market price differential ($) - Put - 07/04/16

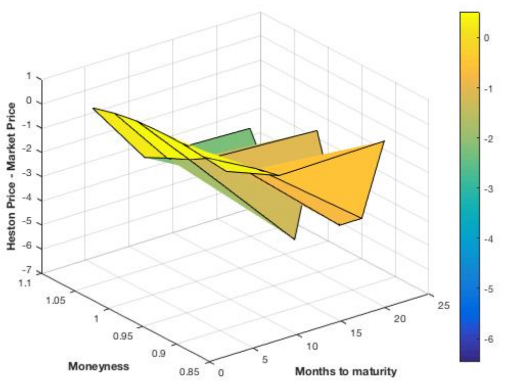


Figure 77: Goldman Sachs SABR - Market price differential ($) - Put - 07/04/16

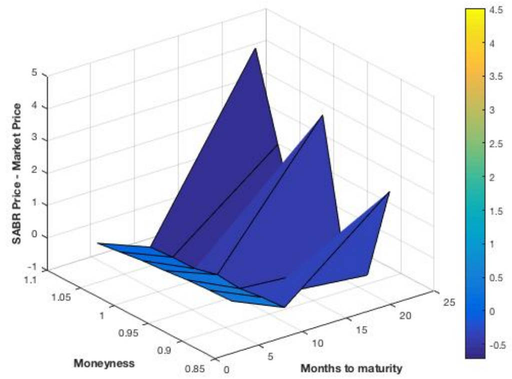
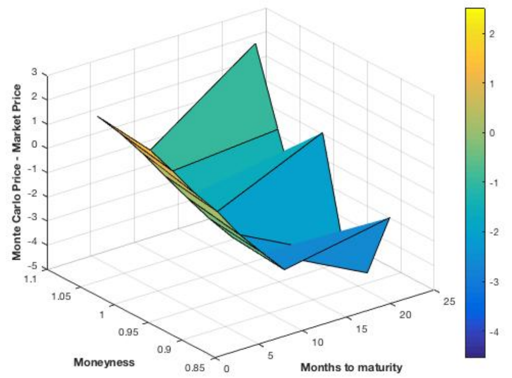


Figure 78: Goldman Sachs Monte Carlo - Market price differential ($) - Put - 07/04/16



In terms of the price differential (Figures 76 to 78), the three methodologies are quite precise for when the option has a few months to maturity. The Heston differential indeed approximates \$0, the SABR reaches \$0 as well and the Monte Carlo methodology too.

For longer maturities, the Monte Carlo model is slightly more precise, as the absolute value in differentials does not go beyond \$3, while for the Heston model it goes to \$6 for the put being in-the-money with two years to maturity and while for the SABR model the price differential is above \$4 for the put having two years to maturity and being in-the-money.

8.3.2 Underlying depreciation between 1% and 0%

For this analysis, let us firstly state that the implied volatilities used for the Heston and SABR models have been averaged after stock depreciations of 0.29% and 0.10% occurred respectively on March 14^{th} 2016 and April 6^{th} 2016. Tables 22 to 25 show respectively the average implied volatilities and SABR calibrated parameters for calls and puts for this specific underlying depreciation.

The actual market data has been retrieved after a stock depreciation of 0.09% occurred on April 6^{th} 2016.

Table 22: Goldman Sachs Mean of Implied Volatilities (%) - Call

	Jul/16	Jan/17	Jan/18
150	26.74	25.87	26.08
155	25.62	25.10	26.44
160	24.61	24.47	26.61
165	23.71	23.89	27.30
170	23.03	23.41	25.01
175	22.62	22.96	25.13
180	22.33	22.52	25.48

Table 23: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
Jul/16	0.31	0.5	-0.44	1.15
Jan/17	0.31	0.5	-0.43	0.65
Jan/18	0.33	0.5	-0.14	0.07

Figure 79: Goldman Sachs Market Implied Volatility (%/100) - Call - 06/04/16

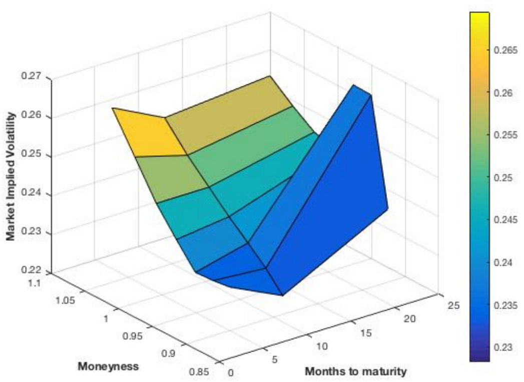


Figure 80: Goldman Sachs Heston Implied Volatility (%/100) - Call - 06/04/16

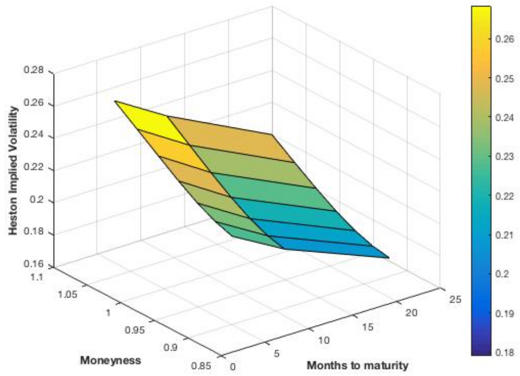


Figure 81: Goldman Sachs SABR Implied Volatility (%/100) - Call - 06/04/16

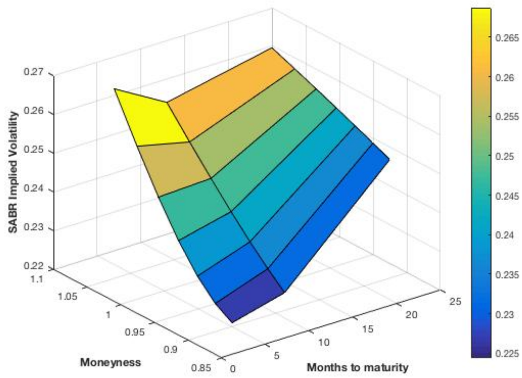
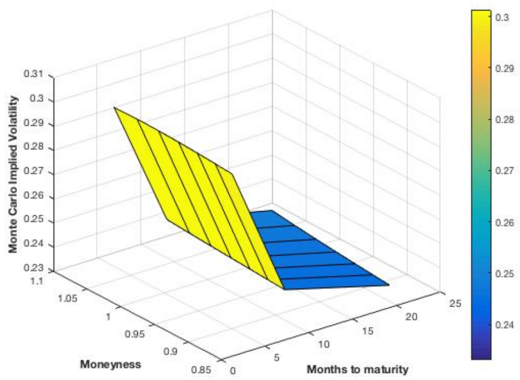


Figure 82: Goldman Sachs Monte Carlo Implied Volatility (%/100) - Call - 06/04/16



As we can observe from Figures 79 to 82, the SABR model is the one that most closely approaches the actual market volatility data. Indeed, except for the spike in volatility occurring for the call being slightly out-of-the-money with two years to maturity, the rest of strikes and maturities points show perfectly estimated skews.

The Heston model does a discrete job too, except that it underestimates implied volatilities when the call is out-of-the-money.

Lastly, the Monte Carlo methodology performs poorly in terms of estimating implied volatility as it overestimates it for the very short time to maturity and fails to show the volatility skews.

Figure 83: Goldman Sachs Heston - Market price differential (\$) - Call - 06/04/16

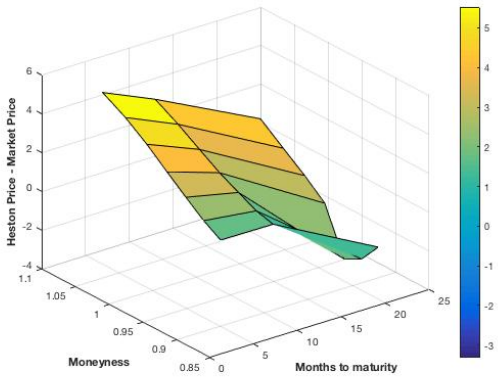


Figure 84: Goldman Sachs SABR - Market price differential (\$) - Call - 06/04/16

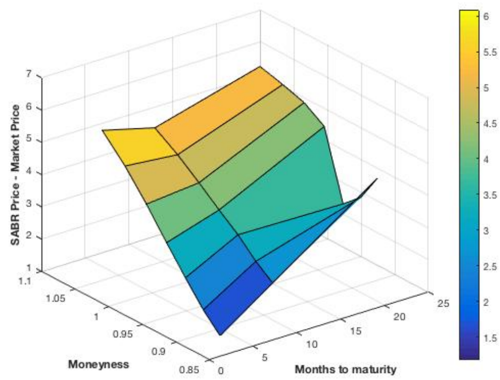
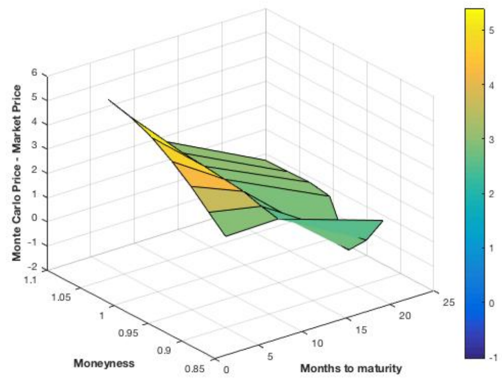


Figure 85: Goldman Sachs Monte Carlo - Market price differential (\$) - Call - 06/04/16



However, in terms of pricing (Figures 83 to 85), the three models can be said to perform equally. Indeed, for in-the-money calls, the biggest discrepancy is shown by the SABR model (\$5 of overpricing vs. \$4 for the other two models), while for out-of-the-money calls the SABR model has a smaller discrepancy (\$1.5 in absolute value) than the Heston and Monte Carlo models (about \$2 for each).

Let us now see the performances of the models with respect of the put option:

Table 24: Goldman Sachs Mean of Implied Volatilities (%) - Put

	Jul/16	Jan/17	Jan/18
130	42.77	29.82	31.94
135	40.05	27.63	28.89
140	33.56	28.11	30.48
145	28.24	27.19	27.68
150	27.02	26.46	26.93
155	25.89	25.72	26.94
160	24.96	25.09	27.10

Table 25: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
Jul/16	0.24	0.5	0.16	3.57
Jan/17	0.32	0.5	-0.48	0.51
Jan/18	0.27	0.5	0.09	1.28

Figure 86: Goldman Sachs Market Implied Volatility (%/100) - Put - 06/04/16

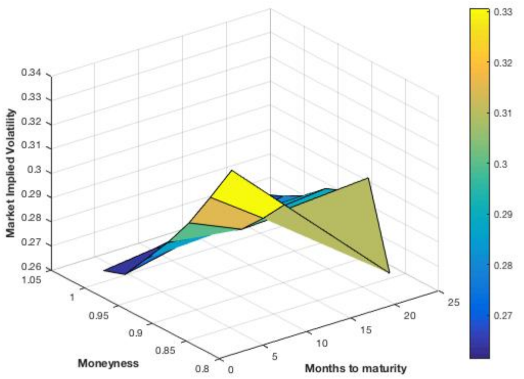


Figure 89: Goldman Sachs Monte Carlo Implied Volatility (%/100) - Put - 06/04/16

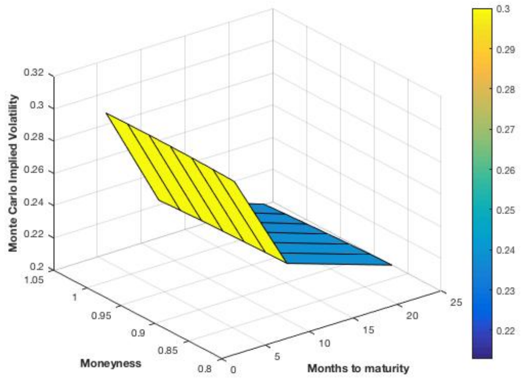
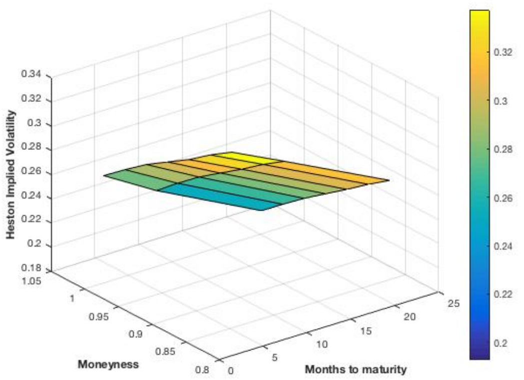


Figure 87: Goldman Sachs Heston Implied Volatility (%/100) - Put - 06/04/16



The SABR model (figure 88) seems to be the winner in terms of estimating implied volatility. Indeed, it successfully shows the spike in volatility when the put has a few months to maturity and is out-of-the-money. Moreover, it captures the skews in volatility correctly for all maturities. Lastly, the estimations of implied volatility are the closest when comparing them to those of the other two models.

The Heston model (figure 87) underestimates volatility at every strike and maturity, and the same can be stated for the Monte Carlo methodology (figure 89). Moreover, they both do not properly show the skews shown by the market data.

Figure 88: Goldman Sachs SABR Implied Volatility (%/100) - Put - 06/04/16

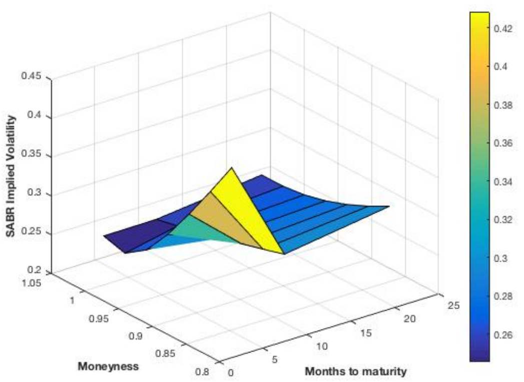


Figure 90: Goldman Sachs Heston - Market price differential ($) - Put - 06/04/16

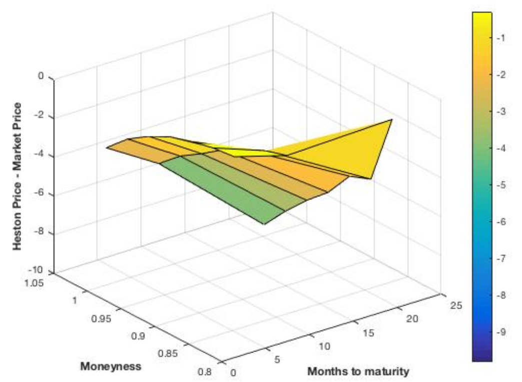


Figure 91: **Goldman Sachs SABR - Market price differential ($) - Put - 06/04/16**

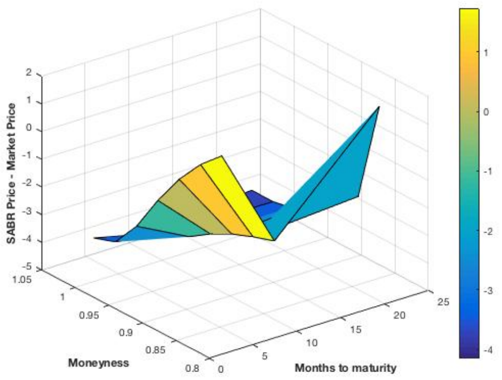
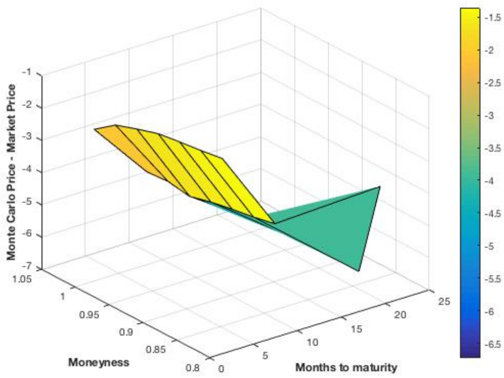


Figure 92: Goldman Sachs Monte Carlo - Market price differential ($) - Put - 06/04/16



With respect to the pricing of the put, it is interesting to point out that the three models underprice the option at almost all strikes and maturities. The Heston model (figure 90) underestimates the put by about $1 for all maturities and strikes, except when the put is out-of-the-money with a few months to maturity, while the Monte Carlo process (figure 92) does so by $1.5 for short maturities and by $4 for longer ones.

The SABR model (figure 91) underestimates the put when it is in-the-money by about $3. When the option is out-of-the-money, the price differentials approach the $0 figure.

Therefore, because of the previous analysis, the Heston model can be declared as the best model to use for pricing puts in this particular instance.

8.3.3 Underlying appreciation between 0% and 1%

For this section, we will analyse the three models after underlying appreciations of 0.55% and 0.95% occurred respectively on March 28^{th} 2016 and March 30^{th} 2016. Please refer to Tables 26 to 29 for the average implied volatilities and SABR calibrated parameters used to estimate the implied volatilities and options prices.

For the market data, the stock appreciation has been of 0.66% and has occurred on April 14^{th} 2016.

Table 26: Goldman Sachs Mean of Implied Volatilities (%) - Call

	Jul/16	Jan/17	Jan/18
150	25.95	25.74	25.90
155	24.56	24.69	24.66
160	23.66	24.11	25.45
165	22.82	23.65	25.09
170	22.11	23.29	24.43
175	21.55	22.26	24.22
180	21.52	22.05	23.42

Table 27: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
Jul/16	0.29	0.5	-0.39	1.23
Jan/17	0.30	0.5	-0.49	0.59
Jan/18	0.32	0.5	-0.73	0.13

Figure 93: Goldman Sachs Market Implied Volatility (%/100) - Call - 14/04/16

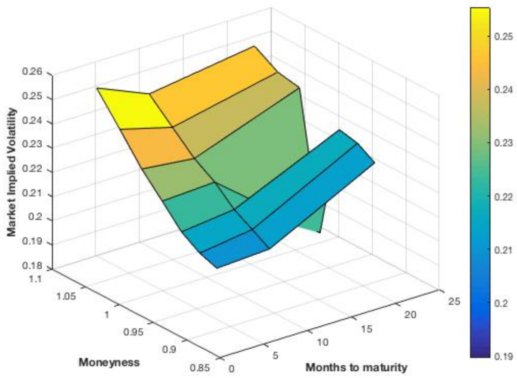


Figure 94: Goldman Sachs Heston Implied Volatility (%/100) - Call - 14/04/16

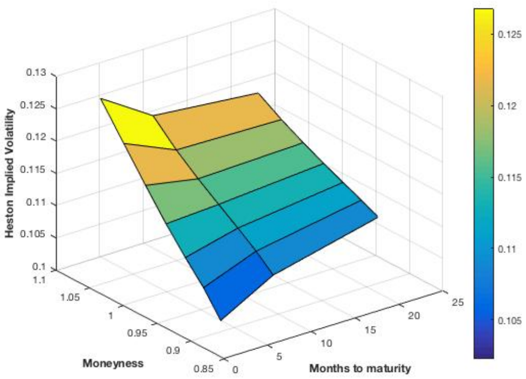


Figure 95: Goldman Sachs SABR Implied Volatility (%/100) - Call - 14/04/16

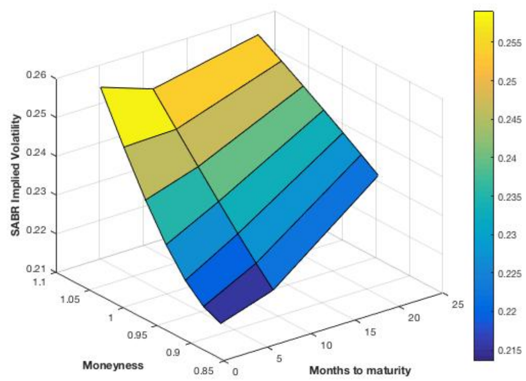
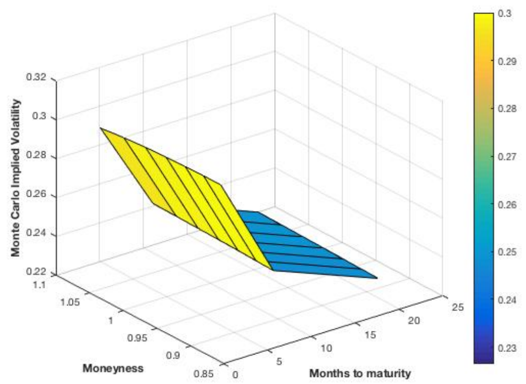


Figure 96: Goldman Sachs Monte Carlo Implied Volatility (%/100) - Call - 14/04/16



While the Heston model (Figure 94) correctly portrays skews for all maturities, it underestimates the implied volatility greatly.

Instead, the SABR model (Figure 95) is more precise and calibrates the values of implied volatilities correctly, except for the low value in implied volatility registered for the call having two years to maturity and being at-the-money.

The Monte Carlo model (Figure 96), once again, fails to show the skews for the various maturities, and overestimates volatility for the option having one month to maturity.

Figure 97: Goldman Sachs Heston - Market price differential ($) - Call - 14/04/16

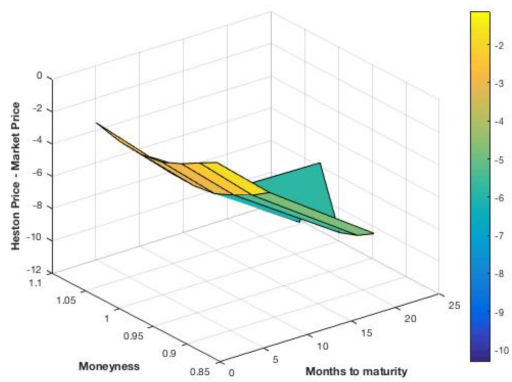


Figure 98: Goldman Sachs SABR - Market price differential ($) - Call - 14/04/16

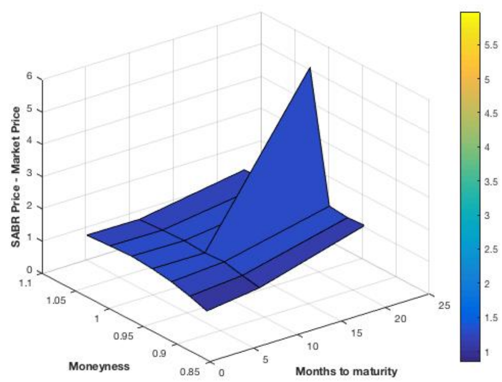
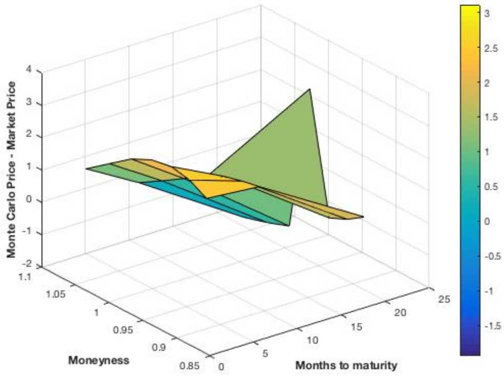


Figure 99: Goldman Sachs Monte Carlo - Market price differential ($) - Call - 14/04/16



Also in terms of pricing the call, the SABR model (Figure 98) is the winner. Indeed, it overprices the option by about $1 for all maturities and strikes, except for an overpricing of $5 when the call has two years to maturity and is at-the-money.

The Heston model (Figure 97), on the other hand, underprices the call by over $2 for short maturities and by over $4 for longer ones.

Lastly, the Monte Carlo process (Figure 99) overprices the call between $1.5 and $2 for all maturities and strikes.

Table 28: Goldman Sachs Mean of Implied Volatilities (%) - Put

	Jul/16	Jan/17	Jan/18
130	32.74	30.83	30.68
135	30.77	29.41	28.81
140	29.96	28.50	28.89
145	27.69	27.59	28.50
150	26.49	27.32	30.10
155	25.38	26.78	27.58
160	24.63	26.19	27.14

Table 29: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
Jul/16	0.30	0.5	-0.23	1.40
Jan/17	0.31	0.5	0.02	1.09
Jan/18	0.35	0.5	-0.44	0.21

Figure 100: Goldman Sachs Market Implied Volatility (%/100) - Put - 14/04/16

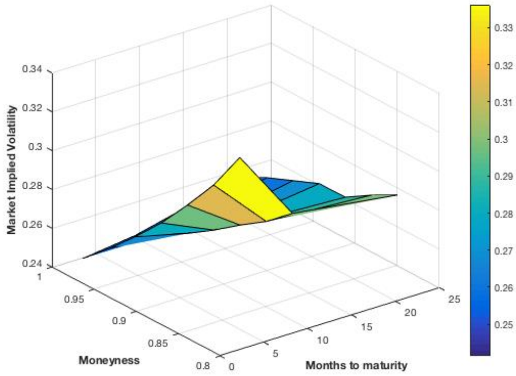


Figure 101: Goldman Sachs Heston Implied Volatility (%/100) - Put - 14/04/16

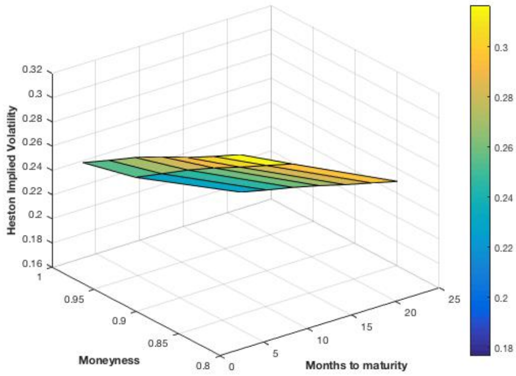


Figure 102: Goldman Sachs SABR Implied Volatility (%/100) - Put - 14/04/16

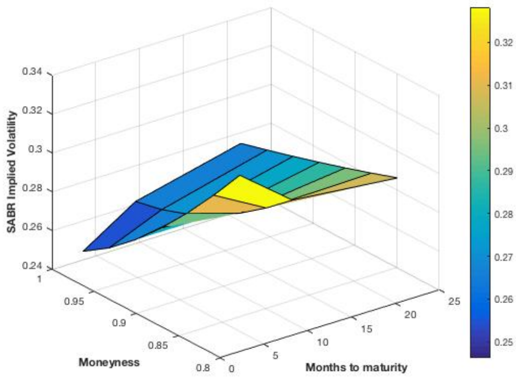
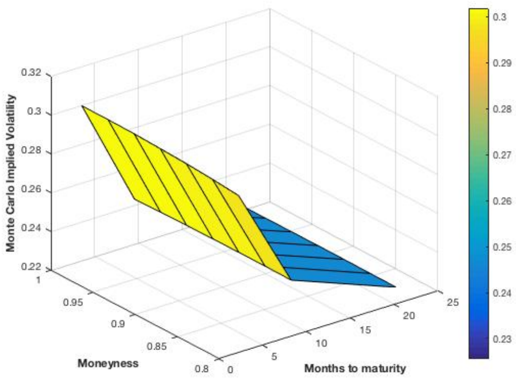


Figure 103: Goldman Sachs Monte Carlo Implied Volatility (%/100) - Put - 14/04/16



When analysing the put, the SABR model (Figure 102) is the only one that manages to catch the spike in volatility when the option is out-of-the-money with a few months to maturity. Moreover, it calibrates the values of implied volatilities at all strikes and maturities.

The Heston model (Figure 101) does calibrate them correctly too, but does not show the famous spike that has been mentioned in the previous sentence.

The Monte Carlo calibration process (Figure 103) overestimates volatility for short maturities and does the opposite for longer ones, as it has often happened in other circumstances.

Figure 104: Goldman Sachs Heston - Market price differential ($) - Put - 14/04/16

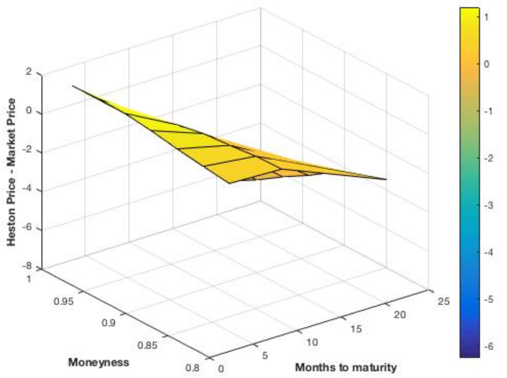


Figure 105: Goldman Sachs SABR - Market price differential ($) - Put - 14/04/16

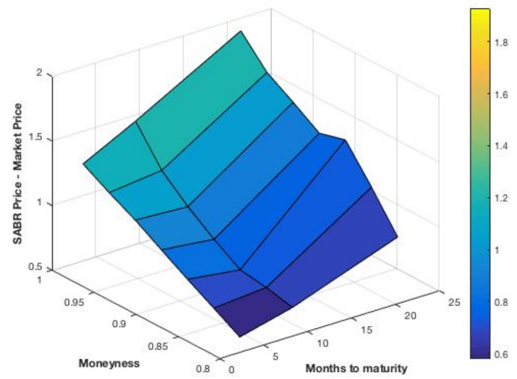
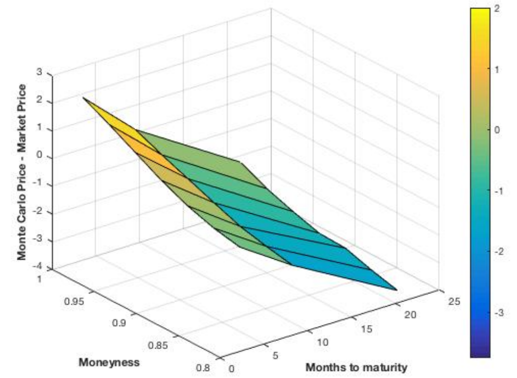


Figure 106: Goldman Sachs Monte Carlo - Market price differential ($) - Put - 14/04/16



And in terms of pricing the put the SABR model (Figure 105) is the winner. Indeed, it overprices the option by only $1 for short maturities and $2 for longer ones, vs. the correct pricing of short-dates options Heston model (Figure 104), which however underprices the options with longer maturities by about $4.

The Monte Carlo process (Figure 106) also underestimates prices for long-dated puts by about $4 and by about $1 for short-dated ones.

8.3.4 Underlying appreciation between 1% and 5%

Here the analysis comes after two stock appreciations of 1.81% and 1.28% occurred on April 1^{st} 2016 and April 11^{th} 2016 respectively. For the actual market data, it has been retrieved after a stock appreciation of 2.29% occurred on April 19^{th} 2016. Tables 30 to 33 show the average implied volatilities and SABR calibrated parameters with respect to these specific stock percentage increases.

Table 30: Goldman Sachs Mean of Implied Volatilities (%) - Call

	Jul/16	Jan/17	Jan/18
150	25.89	25.24	25.26
155	24.68	24.46	22.90
160	23.67	23.93	25.06
165	22.75	23.32	24.27
170	22.00	22.79	24.99
175	21.72	22.37	23.02
180	21.51	21.97	23.83

Table 31: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
Jul/16	0.29	0.5	-0.35	1.26
Jan/17	0.30	0.5	-0.39	0.65
Jan/18	0.31	0.5	0.18	0.08

Figure 107: Goldman Sachs Market Implied Volatility (%/100) - Call - 19/04/16

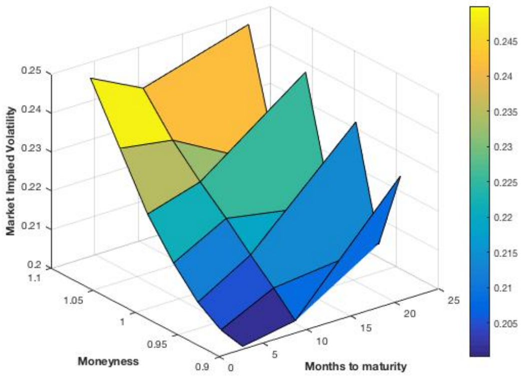


Figure 108: Goldman Sachs Heston Implied Volatility (%/100) - Call - 19/04/16

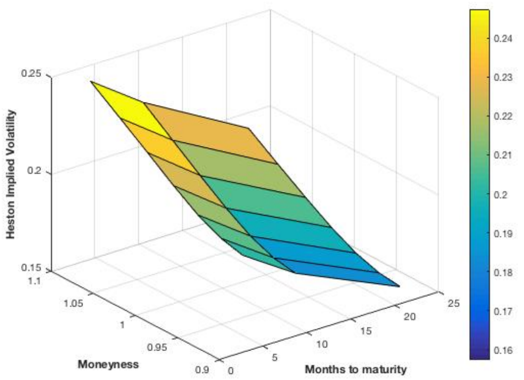


Figure 109: Goldman Sachs SABR Implied Volatility (%/100) - Call - 19/04/16

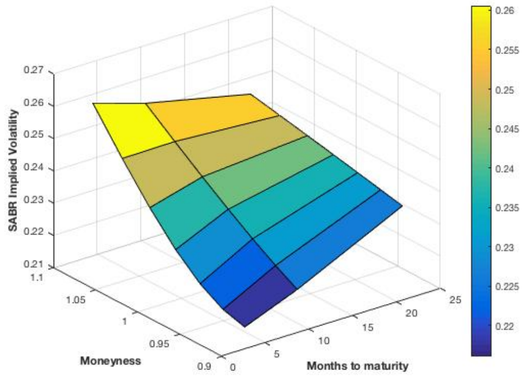
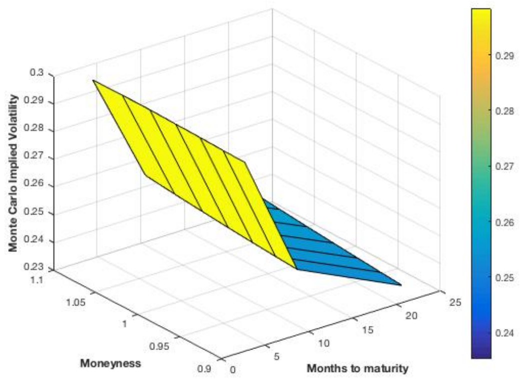


Figure 110: Goldman Sachs Monte Carlo Implied Volatility (%/100) - Call - 19/04/16



The market implied volatility data (Figure 107) shows spikes in volatility for long-dated calls, and a more uniform skew for short-dates ones. Respective skews can be noticed in both the Heston model ((Figure 108)which however underestimates the call implied volatility as the option becomes out-of-the-money) and the SABR model ((Figure 109)which shows levels of implied volatility closer to those of the market).

The Monte Carlo process (Figure 110) not only overestimates the call implied volatilities when it has a few months to maturity, but also fails to show such skews.

Figure 111: Goldman Sachs Heston - Market price differential (\$) - Call - 19/04/16

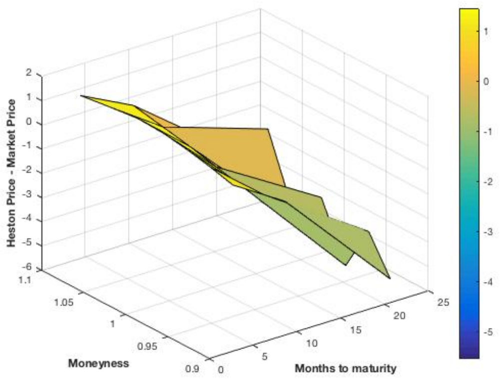


Figure 112: Goldman Sachs SABR - Market price differential (\$) - Call - 19/04/16

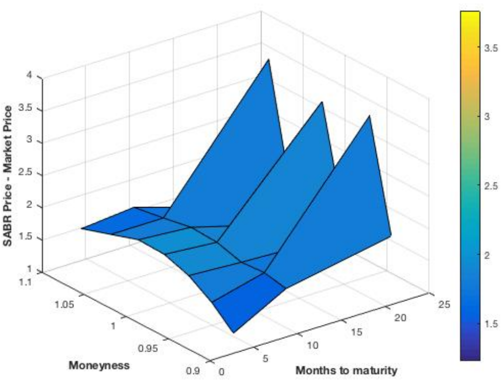
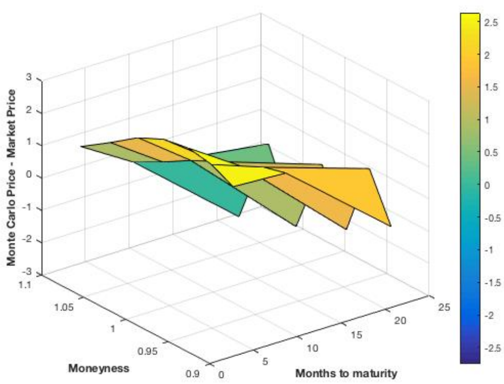


Figure 113: Goldman Sachs Monte Carlo - Market price differential (\$) - Call - 19/04/16



Surprisingly enough, for the pricing of the call (Figures 111 to 113), the Monte Carlo process is the most precise one. Indeed, its price differentials to not go beyond the absolute value of \$2, while for the Heston model the price differential reaches the -\$4 figure for out-of-the-money calls and for the SABR model the differential arrives to \$3.5 for long-dated calls.

Let us now see how the three construction methodologies compare with respect to the put option:

Table 32: Goldman Sachs Mean of Implied Volatilities (%) - Put

	Jul/16	Jan/17	Jan/18
130	32.93	30.44	30.04
135	31.15	29.40	29.59
140	29.47	28.52	28.77
145	28.12	27.58	28.14
150	26.76	26.89	30.28
155	25.67	26.11	27.06
160	24.68	26.73	26.89

Table 33: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
Jul/16	0.30	0.5	-0.20	1.40
Jan/17	0.31	0.5	0.17	1.27
Jan/18	0.35	0.5	-0.79	0.16

Figure 114: Goldman Sachs Market Implied Volatility (%/100) - Put - 19/04/16

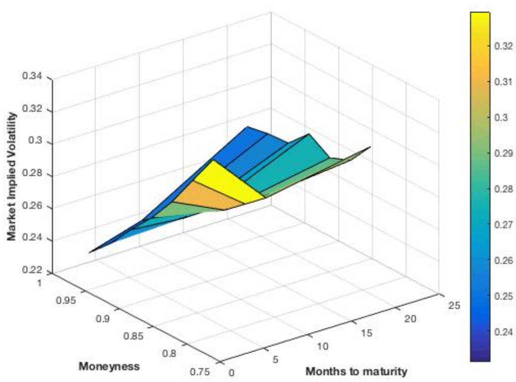


Figure 115: Goldman Sachs Heston Implied Volatility (%/100) - Put - 19/04/16

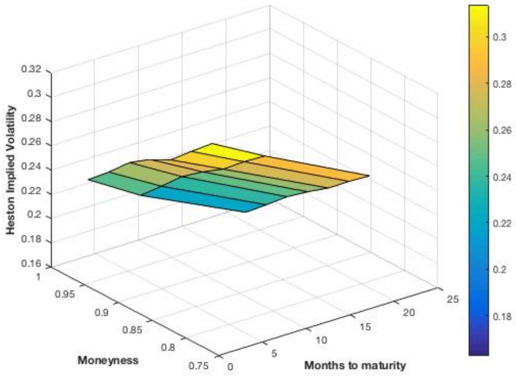


Figure 116: Goldman Sachs SABR Implied Volatility (%/100) - Put - 19/04/16

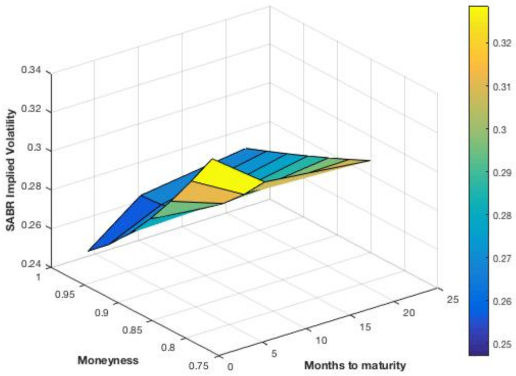
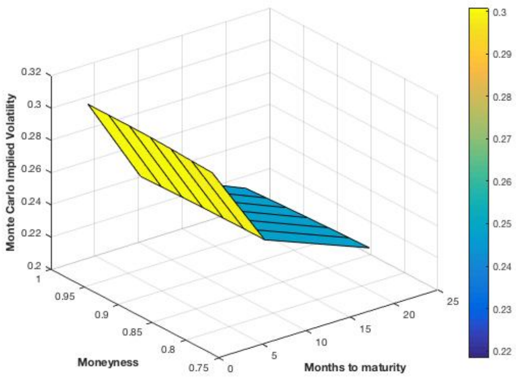


Figure 117: Goldman Sachs Monte Carlo Implied Volatility (%/100) - Put - 19/04/16



It can be seen that the SABR model (Figure 116) is the one which approximates implied volatility the best. It captures the spike in volatility when the put is out-of-the-money with a few months to maturity, and shows correctly the skews for all maturities.

The Heston model (Figure 115) manages to show skews too, but it underestimates implied volatility slightly at all strikes and maturities.

The Monte Carlo method (Figure 117) overestimates volatility for short-dated puts and does the opposite for long-dated ones and fails to show skews for the various times to maturity.

Figure 118: Goldman Sachs Heston - Market price differential ($) - Put - 19/04/16

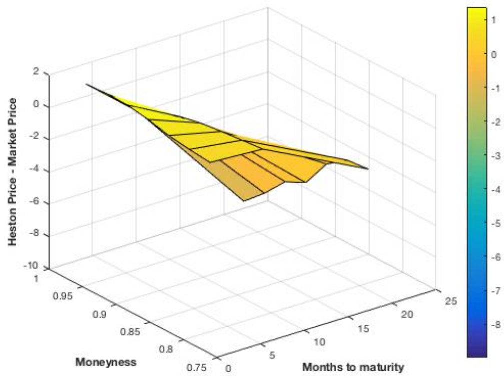


Figure 119: Goldman Sachs SABR - Market price differential ($) - Put - 19/04/16

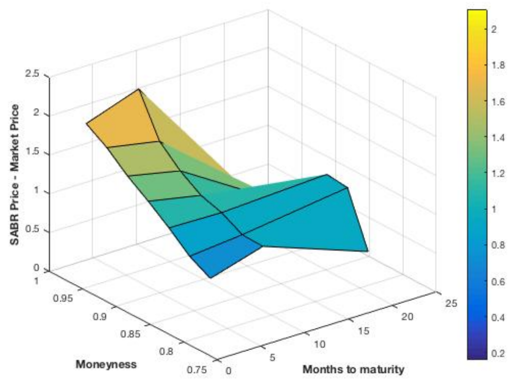
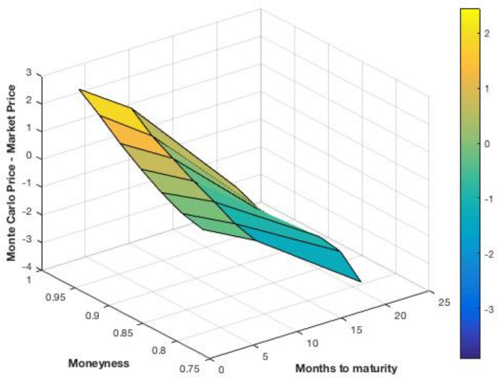


Figure 120: Goldman Sachs Monte Carlo - Market price differential ($) - Put - 19/04/16



Lastly, in terms of the pricing of the put, we can see that the Heston model (Figure 118) correctly prices the option for short maturities and underestimates its price for longer maturities, up to an absolute value of $8.

The SABR model (Figure 119) has a ranged over-pricing of the call between $2 and $0.5.

The Monte Carlo (Figure 120) method's ranges vary from $2 to -$4.

Hence, the SABR model is the most stable one of the three, with also the lowest average in absolute value of price differential.

8.4 Zions Bancorporation

Zions Bancorporation is a mid-market capitalisation US bank ($4.54 billion[216]) with headquarters in Salt Lake Citi, Utah, and employs about 10000 people[217].

It has been chosen as a validate comparable company for this study because there is a lot of data available on the Internet concerning its option prices and implied volatilities (all of the following results are based on strikes, maturities, market prices and implied volatilities found at the following footnote[218]).

Moreover, the way Zions Bancorporation should be viewed different from Citigroup is exclusively in the market capitalisation.

A dividend of $0.24 per share[219] has been considered in the following calculations as well.

Let us now show how the Heston, SABR and Monte Carlo models have performed with respect to the market implied volatilities and prices.

[216] http://finance.yahoo.com/q?s=ZION
[217] https://en.wikipedia.org/wiki/Zions Bancorporation
[218] http://finance.yahoo.com/q/op?s=ZION+Options
[219] http://finance.yahoo.com/q?s=ZION

8.4.1 Underlying depreciation between 5% and 1%

For this analysis, the average implied volatilities have been retrieved after a double stock depreciation occurred on both March 31st 2016 (-1.71%) and on April 5th 2016 (-1.99%). The respective average implied volatilities surfaces and SABR calibrated parameters for calls and puts can be found at Tables 34 to 37. The actual market data and implied volatility has been obtained on April 7th 2016 after an underlying depreciation of 2.80%.

Let us firstly see how the three models compare to each other for the call option:

Table 34: Zions Bancorporation Mean of Implied Volatilities (%) - Call

	Jul/16	Jan/17	Jan/18
22	58.60	54.03	39.11
23	54.35	44.57	40.44
24	35.45	42.73	35.93
25	46.69	40.90	31.42
26	32.15	39.09	30.94
27	31.18	37.28	30.47
28	30.62	34.50	30.85
29	30.47	32.50	30.85
30	34.86	31.16	32.58

Table 35: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
Jul/16	0.25	0.5	-0.84	3.67
Jan/17	0.23	0.5	-0.77	1.78
Jan/18	0.15	0.5	-0.58	1.81

Figure 121: Zions Bancorporation Market Implied Volatility (%/100) - Call - 07/04/16

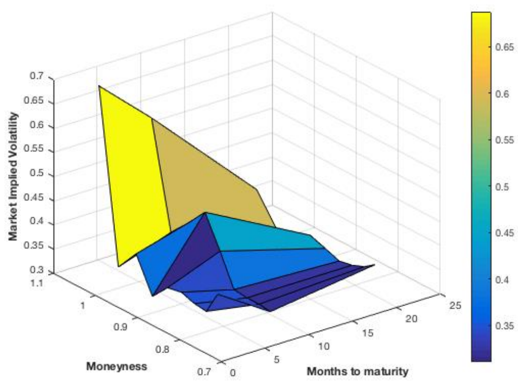


Figure 122: Zions Bancorporation Heston Implied Volatility (%/100) - Call - 07/04/16

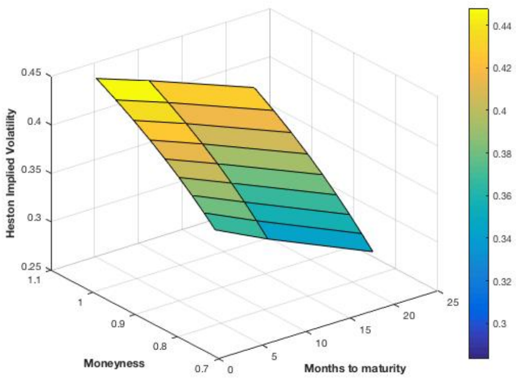


Figure 123: Zions Bancorporation SABR Implied Volatility (%/100) - Call - 07/04/16

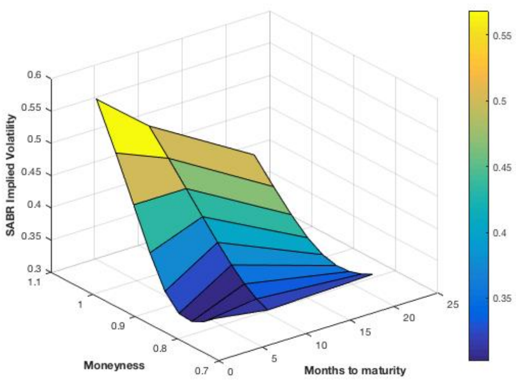
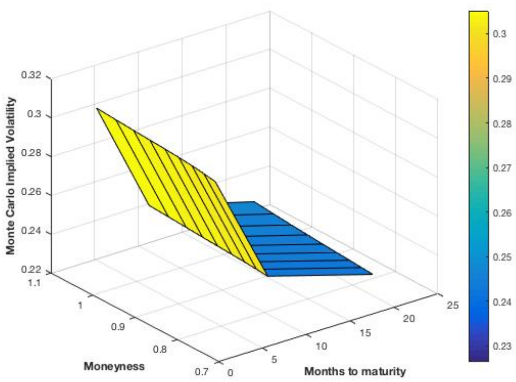


Figure 124: Zions Bancorporation Monte Carlo Implied Volatility (%/100) - Call - 07/04/16



As we can see, the SABR model (Figure 123) is the one that calibrates most precisely the implied volatility with respect to the market. Indeed, it manages to capture the spike in volatility when the option is very in-the-money and with one month to maturity. However, it shows skews for all maturities, something that is not seen in the market implied volatility surface.

The same can be said for the Heston mode (Figure 122)l: not only does it show skews at all maturities, but it also underestimates volatility at all strikes and maturities. However, it is great in showing the decreasing volatility for the option going form being in-the-money to out-of-the-money.

The Monte Carlo method (Figure 124) underprices volatility at all strikes and maturities, and fails to show the bumps in volatility shown by the market data.

Figure 125: Zions Bancorporation Heston - Market price differential ($) - Call - 07/04/16

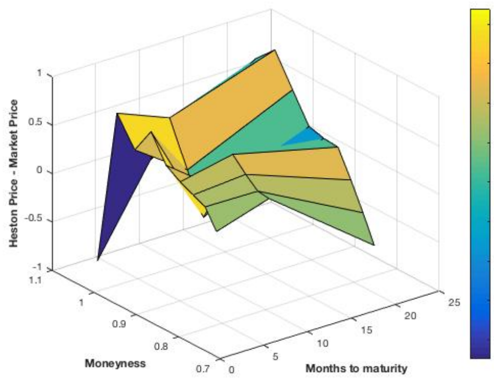


Figure 126: Zions Bancorporation SABR - Market price differential ($) - Call - 07/04/16

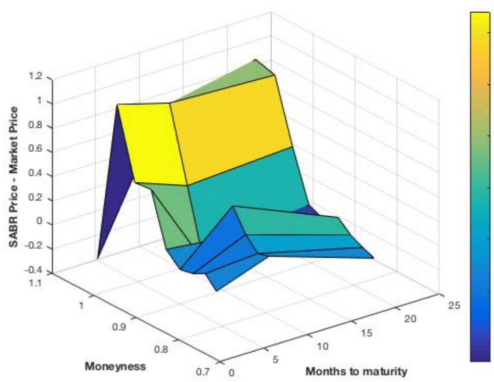
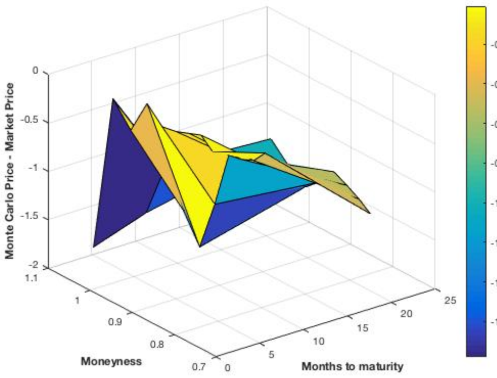


Figure 127: Zions Bancorporation Monte Carlo - Market price differential ($) - Call - 07/04/16



When comparing the models in terms of the pricing of the call (Figures 125 to 127), the Heston and Monte Carlo models do a better job the the SABR model when the call is in-the-money, as the absolute value of the mispricing is of only $0.5 for them and $1 for the SABR model. However, when the option is out-of-the-money, the Heston model and SABR model improve their pricing calibrations, with mispricings approaching the $0 value.

The Monte Carlo model on the other hand still shows mispricings of about $0.5. Therefore, the winner in the comparison is the Heston model.

Table 36: Zions Bancorporation Mean of Implied Volatilities (%) - Put

	Jul/16	Jan/17	Jan/18
15	54.20	45.71	39.35
18	43.27	39.19	35.71
20	38.55	34.21	35.35
23	32.18	32.68	30.27
25	18.97	28.41	27.20
27	28.28	23.26	30.03

Table 37: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
Jul/16	0.13	0.5	-0.46	1.74
Jan/17	0.15	0.5	-1.00	0.61
Jan/18	0.14	0.5	-0.33	0.55

Figure 128: Zions Bancorporation Market Implied Volatility (%/100) - Put - 07/04/16

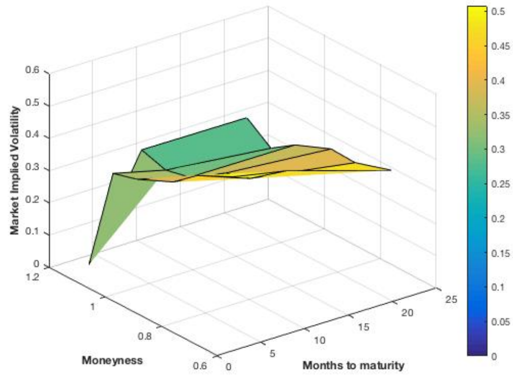


Figure 129: Zions Bancorporation Heston Implied Volatility (%/100) - Put - 07/04/16

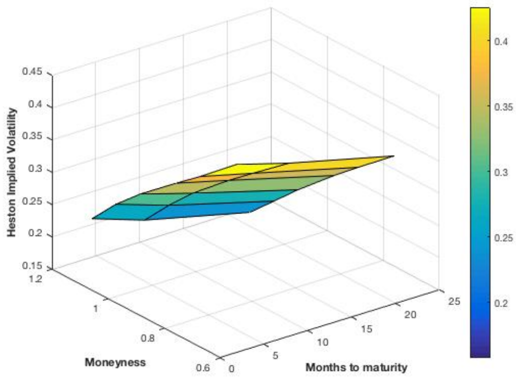


Figure 130: Zions Bancorporation SABR Implied Volatility (%/100) - Put - 07/04/16

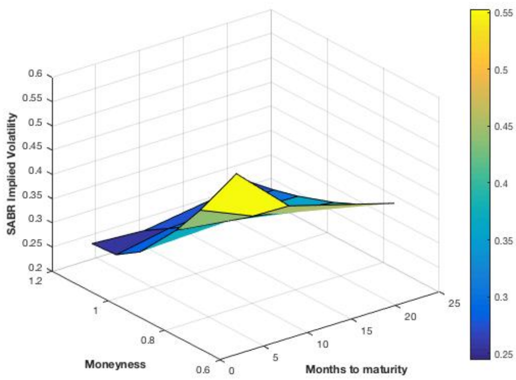
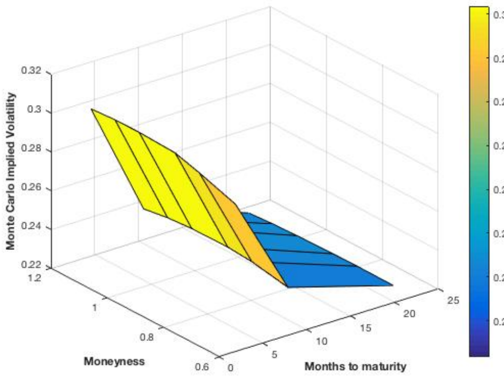


Figure 131: Zions Bancorporation Monte Carlo Implied Volatility (%/100) - Put - 07/04/16



As it can be noticed, the market implied volatility surface (Figure 128) shows skews, thing that can be also seen in the Heston (Figure 129) and SABR (Figure 130) models. Moreover, volatility is correctly estimated correctly from both models (with the SABR winning because it is slightly more precise in the calibration).

The Monte Carlo model (Figure 131), on the other hand, not only underestimates implied volatility at all strikes and maturities, but also fails to portray the skews for the various times to maturity.

Figure 132: Zions Bancorporation Heston - Market price differential ($) - Put - 07/04/16

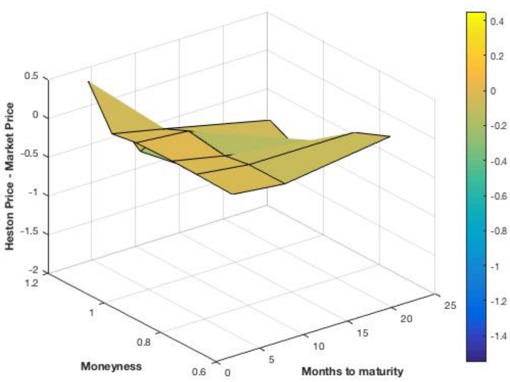


Figure 133: Zions Bancorporation SABR - Market price differential ($) - Put - 07/04/16

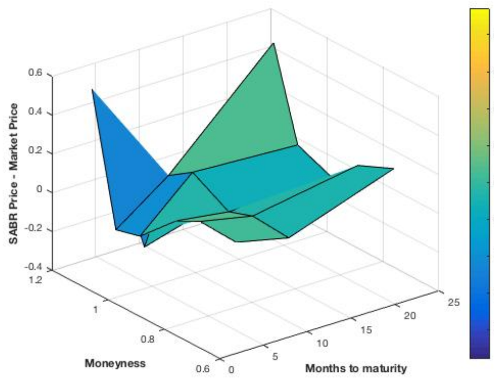
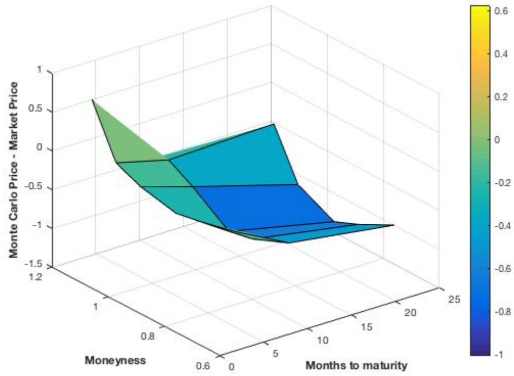


Figure 134: Zions Bancorporation Monte Carlo - Market price differential ($) - Put - 07/04/16



When comparing the three models in terms of the pricing of the option (Figures 132 to 134), the Heston model is a clear winner: indeed, its mispricings are almost $0 for all strikes and maturities. On the other hand, there are big mispricings for the SABR model, especially when the put is in-the-money (up to $0.4) and for Monte Carlo as well (underpricings of $0.6 when the put is out-of-the-money).

Therefore, when there is an underlying movement between -5% and -1% for a small cap bank, the Heston model seems to be the best model to use out of the three analysed here.

8.4.2 Underlying depreciation between 1% and 0%

In this case the average implied volatilities have been found after a stock depreciation of 0.4% occurred on March 14^{th} 2016 and of 0.17% on April 1^{st} 2016. The market data has been found after the stock has again depreciated by 0.12% on April 4^{th} 2016. Tables 38 to 41 show the mean of implied volatilities and calibrated SABR parameters respectively for calls and puts.

Table 38: Zions Bancorporation Mean of Implied Volatilities (%) - Call

	Jul/16	Jan/17	Jan/18
22	46.15	44.27	36.63
23	43.63	38.69	36.18
24	33.18	37.41	33.48
25	38.38	36.13	30.86
26	30.71	35.00	30.40
27	30.05	33.86	30.31
28	29.11	32.50	30.00
29	28.81	31.50	30.50
30	30.77	30.90	30.48

Table 39: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
Jul/16	0.17	0.5	-0.66	2.37
Jan/17	0.18	0.5	-0.56	1.37
Jan/18	0.13	0.5	-0.44	1.56

Figure 135: Zions Bancorporation Market Implied Volatility (%/100) - Call - 04/04/16

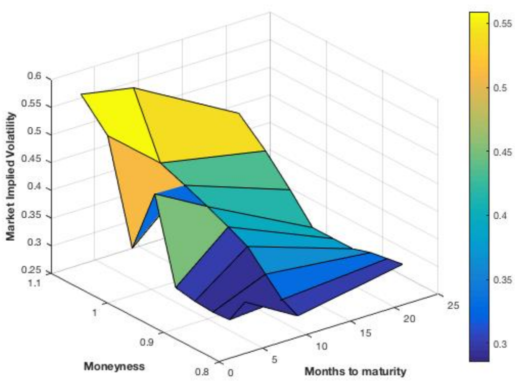


Figure 136: Zions Bancorporation Heston Implied Volatility (%/100) - Call - 04/04/16

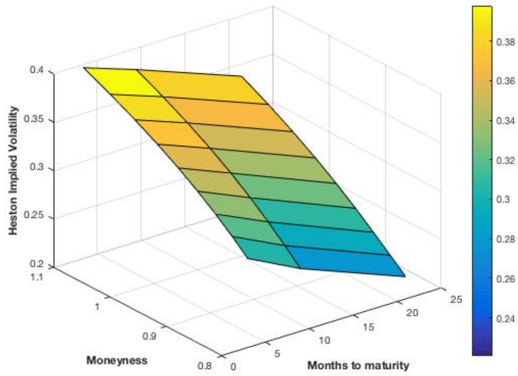


Figure 137: Zions Bancorporation SABR Implied Volatility (%/100) - Call - 04/04/16

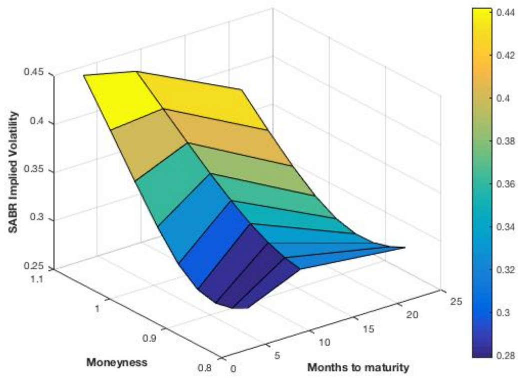
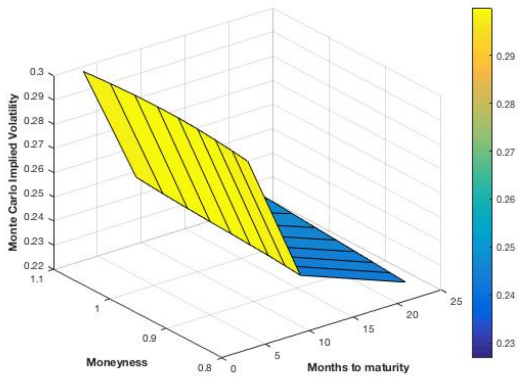


Figure 138: Zions Bancorporation Monte Carlo Implied Volatility (%/100) - Call - 04/04/16



As in the previous case, the SABR model (Figure 137) is the best one to use to capture implied volatility for the call when the underlying movement is negative. Indeed, it captures the skew for all maturities and prices it correctly at all maturities and strikes (except when it underprices at the point where the call is in-the-money with a few months to maturity and except for the spike in volatility when the option is at-the-money, not caught by the model).

The Heston model (Figure 136) also shows correctly skews for all maturities, but has slightly lower levels of volatility than those shown by the market.

The Monte Carlo model (Figure 137), lastly, fails to show such skews and underestimates volatility at all strikes and maturities.

Figure 139: Zions Bancorporation Heston - Market price differential ($) - Call - 04/04/16

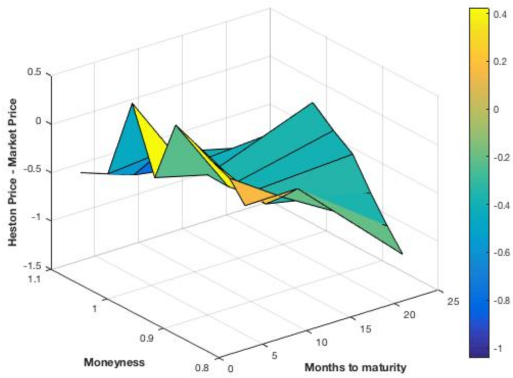


Figure 140: Zions Bancorporation SABR - Market price differential ($) - Call - 04/04/16

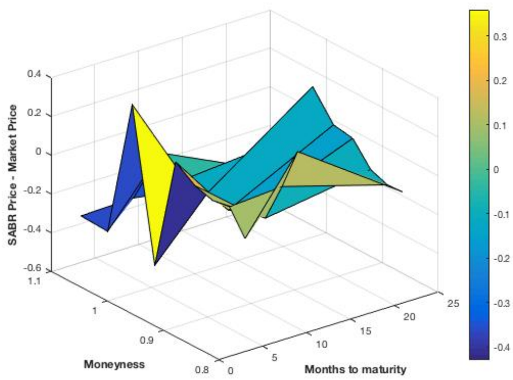
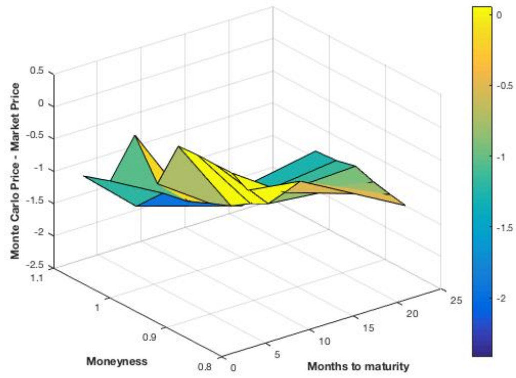


Figure 141: Zions Bancorporation Monte Carlo - Market price differential ($) - Call - 04/04/16



When analysing the pricing of the call (Figures 139 to 141), the Heston model shows underestimations of the price of the option of about $0.3, while the SABR model shows mispricings in absolute value of up to $0.2 and the Monte Carlo model underprices the call at all strikes and maturities between $0.5 and $1. This declares that the SABR model is superior to the other two in this instance.

Let us now see how the methodologies have performed with respect to the put option:

Table 40: Zions Bancorporation Mean of Implied Volatilities (%) - Put

	Jul/16	Jan/17	Jan/18
15	54.50	46.35	40.14
18	43.95	40.16	37.52
20	40.29	36.34	35.75
23	34.67	34.45	30.89
25	27.85	31.35	29.92
27	32.47	28.37	21.67

Table 41: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
Jul/16	0.15	0.5	-0.18	1.58
Jan/17	0.16	0.5	-0.71	0.58
Jan/18	0.15	0.5	-0.12	0.64

Figure 142: Zions Bancorporation Market Implied Volatility (%/100) - Put - 04/04/16

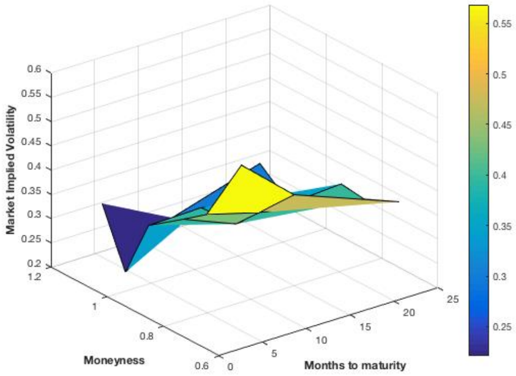


Figure 143: Zions Bancorporation Heston Implied Volatility (%/100) - Put - 04/04/16

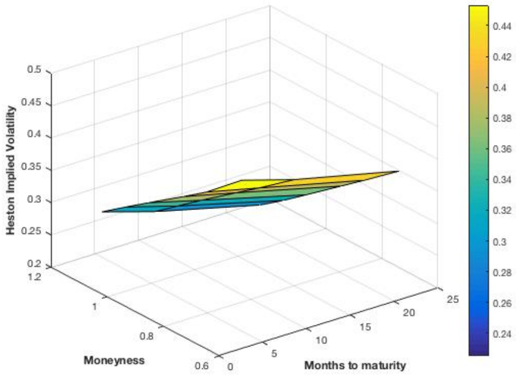


Figure 144: Zions Bancorporation SABR Implied Volatility (%/100) - Put - 04/04/16

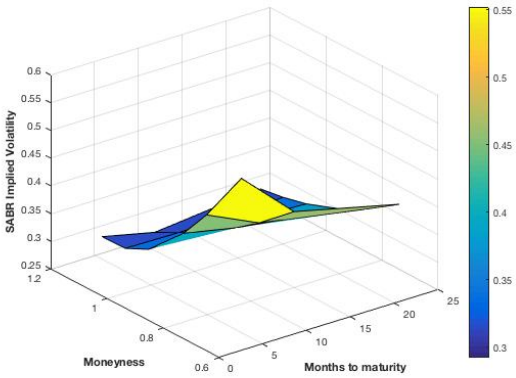
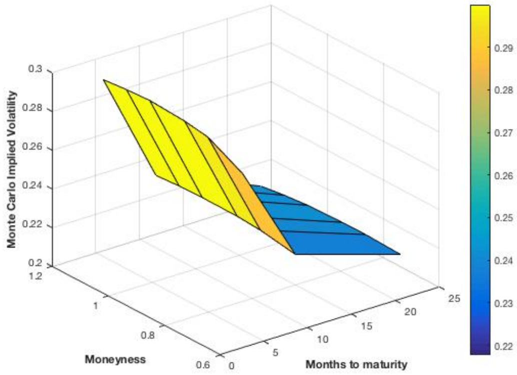


Figure 145: Zions Bancorporation Monte Carlo Implied Volatility (%/100) - Put - 04/04/16



Both the Heston (Figure 143) and SABR (Figure 144) models show skews for the put at all maturities, and they do so correctly (except for a bump in volatility missed out when the option is at-the-money). However, the Heston model underestimates the implied volatility of the put more than does the SABR model.

The Monte Carlo model (Figure 145), once again, fails to show the skews for the various times to maturity, and underestimates volatility at all levels.

Thus, the SABR model is the best model to use in this case.

Figure 146: Zions Bancorporation Heston - Market price differential ($) - Put - 04/04/16

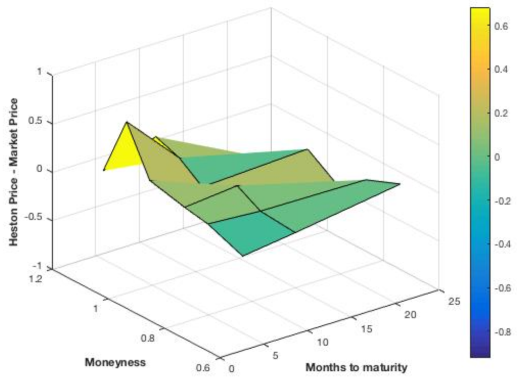


Figure 147: Zions Bancorporation SABR - Market price differential ($) - Put - 04/04/16

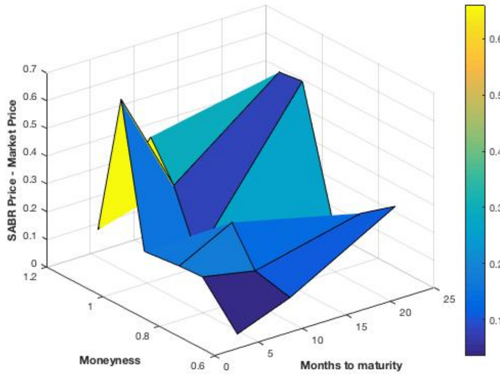
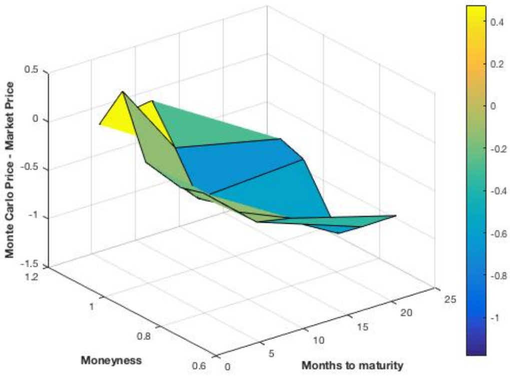


Figure 148: Zions Bancorporation Monte Carlo - Market price differential ($) - Put - 04/04/16



However, even if the SABR model is the winner in estimating implied volatility for this specific case, the same cannot be said when pricing the put (please refer to Figures 146 to 148). Indeed, from the figures we can see that the Heston model misprices the put by up to $0.4, while the SABR model does so up to $0.5, especially when the option is in-the-money. The Monte Carlo method, on the other hand, underprices the put by around $0.5 especially when the option is out-of-the-money.

Therefore, the Heston model wins this particular case.

8.4.3 Underlying appreciation between 0% and 1%

For this study we analyse the implied volatilities through firstly averaging them after stock price appreciations of 0.71% on March 21^{st} 2016 and 0.73% on April 8^{th} 2016, and secondly by recording the market data after an underlying movement of 0.99% occurred on April 14^{th} 2016. Tables 42 to 45 show the average implied volatilities and SABR calibrated parameters for the calls and puts.

Table 42: Zions Bancorporation Mean of Implied Volatilities (%) - Call

	Jul/16	Jan/17	Jan/18
22	51.67	42.00	34.00
23	33.74	34.82	32.48
24	32.11	36.35	32.24
25	31.25	37.88	32.00
26	31.80	34.34	31.19
27	31.52	30.80	30.38
28	30.89	30.25	29.50
29	30.47	29.75	29.00
30	32.57	29.67	28.74

Table 43: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
Jul/16	0.13	0.5	-0.31	2.87
Jan/17	0.17	0.5	-0.56	1.03
Jan/18	0.16	0.5	-0.60	0.30

Figure 149: Zions Bancorporation Market Implied Volatility (%/100) - Call - 14/04/16

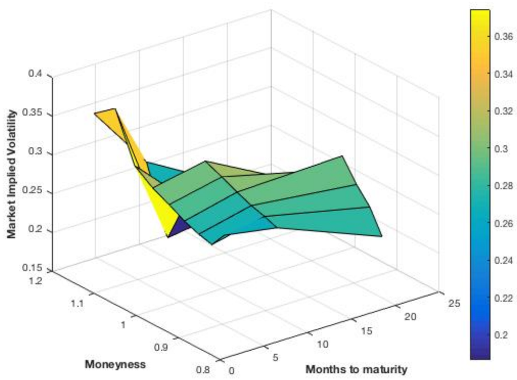


Figure 150: Zions Bancorporation Heston Implied Volatility (%/100) - Call - 14/04/16

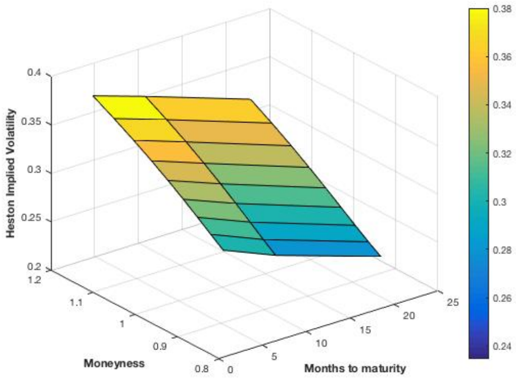


Figure 151: Zions Bancorporation SABR Implied Volatility (%/100) - Call - 14/04/16

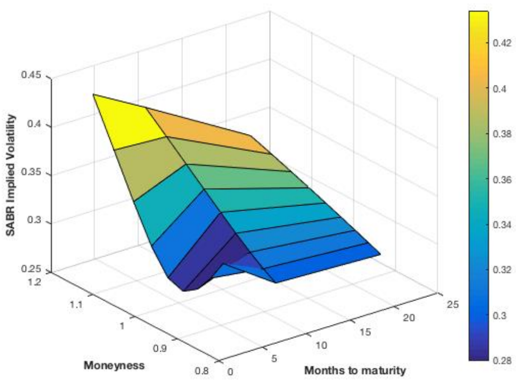
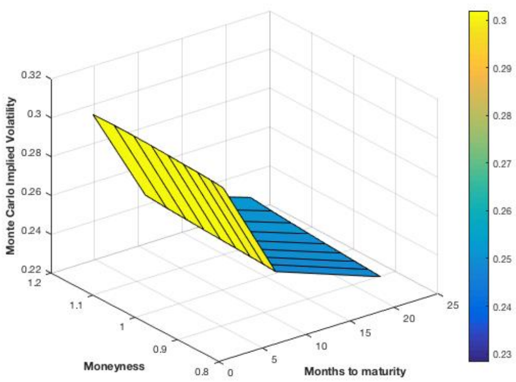


Figure 152: Zions Bancorporation Monte Carlo Implied Volatility (%/100) - Call - 14/04/16



We can see for the call the market implied volatility (Figure 149) stays at about 30% when the call is out-of-the-money, and this is best capture by the Heston model. Moreover, both the Heston (Figure 150) and SABR (Figure 151) model show skews that are not present in the market volatility.

Lastly, the Monte Carlo (Figure 152) methodology once again underestimates volatility at all strikes and maturities, failing to show the spike when the call is in-the-money with a few months to maturity.

Figure 153: Zions Bancorporation Heston - Market price differential ($) - Call - 14/04/16

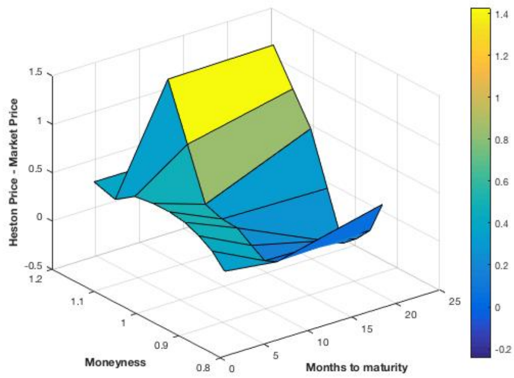


Figure 154: Zions Bancorporation SABR - Market price differential ($) - Call - 14/04/16

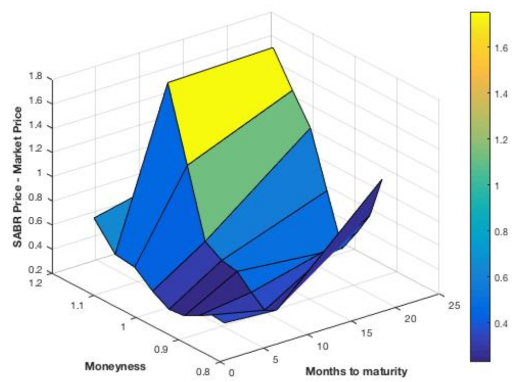
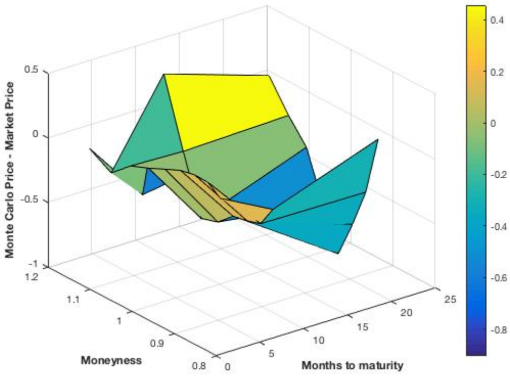


Figure 155: Zions Bancorporation Monte Carlo - Market price differential ($) - Call - 14/04/16



When comparing the three models for the pricing of the call (Figures 153 to 155), the Monte Carlo model is the clear winner. Indeed, its mispricings in absolute value do not pass the $0.5 figure, while for the Heston model there are far bigger mispricings especially when the call is in-the-money.

For the SABR model, the mispricings are over $0.6 for almost all strikes and maturities and pass the $1.5 figure when the option has 2 years to maturity and is in-the-money.

Table 44: Zions Bancorporation Mean of Implied Volatilities (%) - Put

	Jul/16	Jan/17	Jan/18
15	52.93	47.61	40.24
18	46.34	40.33	35.26
20	40.83	36.92	34.75
23	34.50	33.67	31.73
25	32.02	31.11	28.47
27	24.03	31.87	30.82

Table 45: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
Jul/16	0.15	0.5	-1.00	0.94
Jan/17	0.15	0.5	0.05	1.14
Jan/18	0.14	0.5	0.01	0.71

Figure 156: Zions Bancorporation Market Implied Volatility (%/100) - Put - 14/04/16

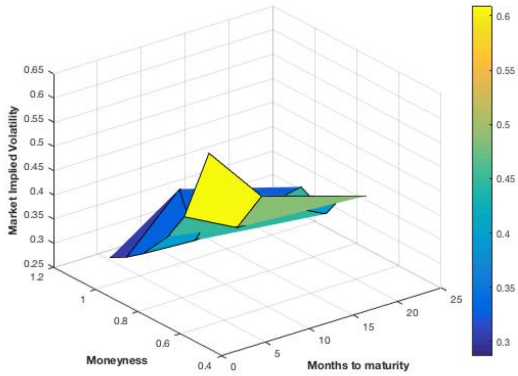


Figure 157: Zions Bancorporation Heston Implied Volatility (%/100) - Put - 14/04/16

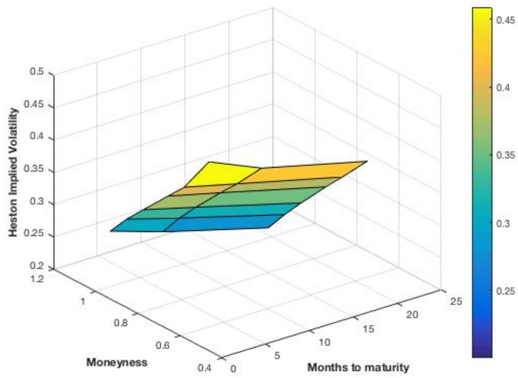


Figure 158: Zions Bancorporation SABR Implied Volatility (%/100) - Put - 14/04/16

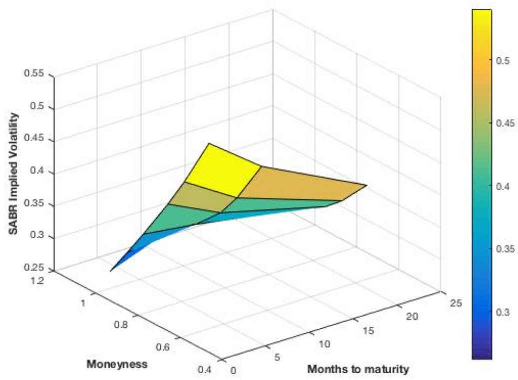
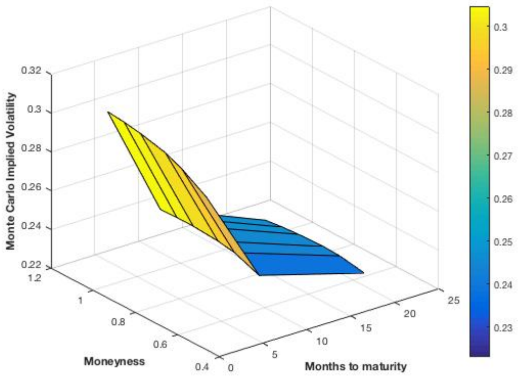


Figure 159: Zions Bancorporation Monte Carlo Implied Volatility (%/100) - Put - 14/04/16



When analysing the put, we can see that both the Heston (Figure 157) and SABR (Figure 159) model do a discrete job at estimating implied volatility. Indeed, they show the skew for the option at all maturities and manage to capture the spike in volatility when the put is far out-of-the-money with a few months to maturity.

The Monte Carlo model (Figure 159), on the other hand, fails once again to show proper implied volatilities by underestimating them at all strikes and maturities and by not showing the skews for the various maturities.

Figure 160: Zions Bancorporation Heston - Market price differential ($) - Put - 14/04/16

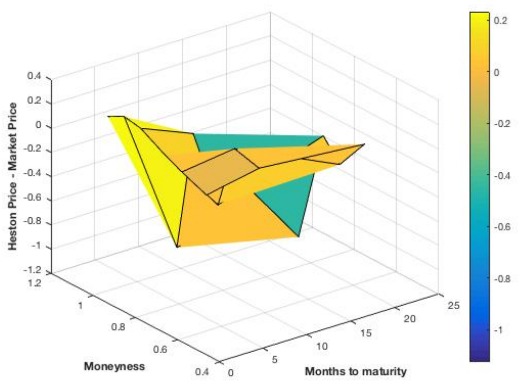


Figure 161: Zions Bancorporation SABR - Market price differential ($) - Put - 14/04/16

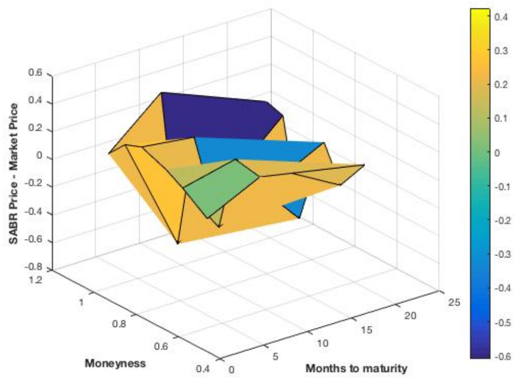
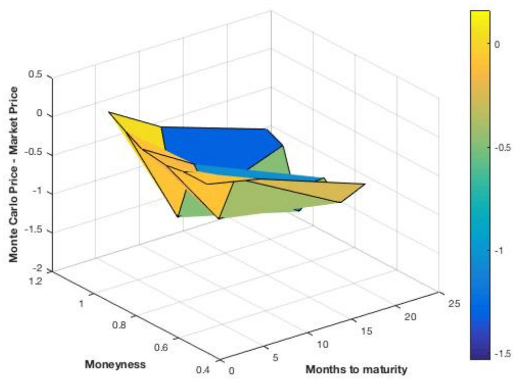


Figure 162: Zions Bancorporation Monte Carlo - Market price differential ($) - Put - 14/04/16



In terms of the price differentials (Figures 160 to 162), we can see that the SABR model is the clear winner. Indeed, its differentials do not go above \$0.4 in absolute value, while for the Heston model greater differentials are seen when the put is at-the-money.

The Monte Carlo methodology compares similarly to the performance of the Heston model by underestimating the price of the put by almost \$1 when the option is at-the-money.

8.4.4 Underlying appreciation between 1% and 5%

For this analysis the averaged implied volatilities have been found after retrieving them from stock price appreciations of respectively 1.23% occurred on March 30^{th} 2016 and of 1.31% on April 6^{th} 2016. For the actual market data, the prices and implied volatilities of calls and puts have been taken after the stock as increased by 1.07% on April 11^{th} 2016.

Tables 46 to 49 show the respective mean of implied volatilities and SABR calibrated parameters for call and put options related to this specific movement in the underlying.

Table 46: Zions Bancorporation Mean of Implied Volatilities (%) - Call

	Jul/16	Jan/17	Jan/18
22	54.49	43.00	36.50
23	49.59	35.63	38.50
24	31.74	37.12	34.94
25	34.94	38.62	31.38
26	29.96	34.98	30.63
27	30.13	31.34	29.88
28	32.25	30.50	30.25
29	29.86	30.00	30.75
30	33.04	29.79	31.43

Table 47: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
Jul/16	0.20	0.5	-0.78	4.01
Jan/17	0.19	0.5	-0.70	0.98
Jan/18	0.14	0.5	-0.47	1.46

Figure 163: Zions Bancorporation Market Implied Volatility (%/100) - Call - 11/04/16

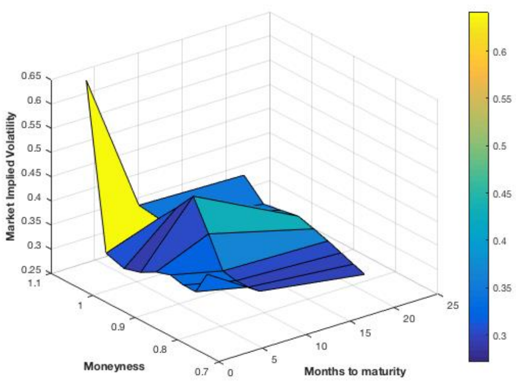


Figure 164: Zions Bancorporation Heston Implied Volatility (%/100) - Call - 11/04/16

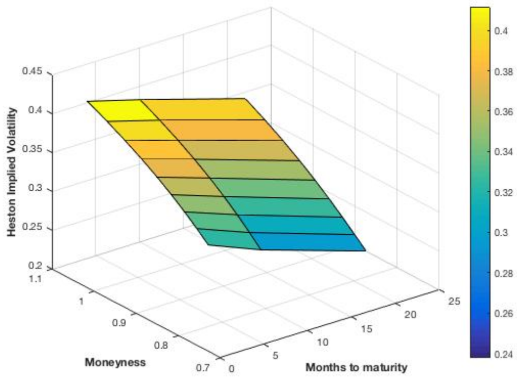


Figure 165: Zions Bancorporation SABR Implied Volatility (%/100) - Call - 11/04/16

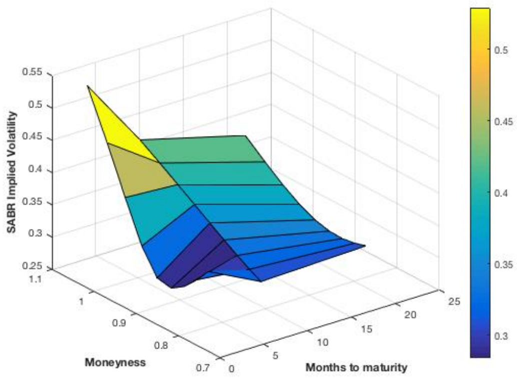
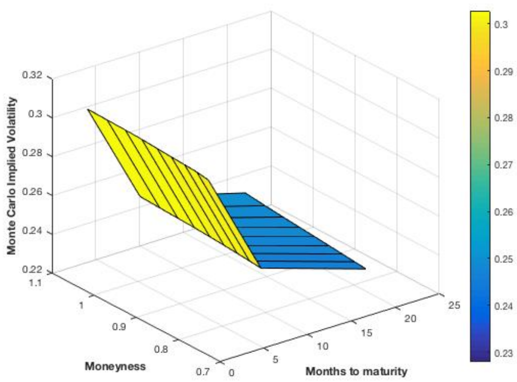


Figure 166: Zions Bancorporation Monte Carlo Implied Volatility (%/100) - Call - 11/04/16



In terms of implied volatilities for the call options (Figures 163 to 166), both the Heston and SABR model to a discrete job: the both correctly estimates the implied volatilities for all strikes and maturities, but they both fail to see the spike in volatility when the option is in-the-money with a few months to maturity.

The Monte Carlo process once again fails to correctly measure implied volatility, underestimating it.

Figure 167: Zions Bancorporation Heston - Market price differential (\$) - Call - 11/04/16

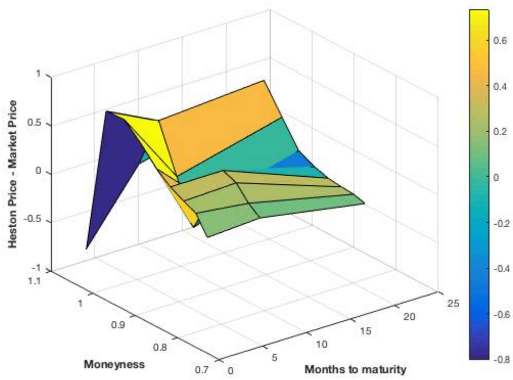


Figure 168: Zions Bancorporation SABR - Market price differential (\$) - Call - 11/04/16

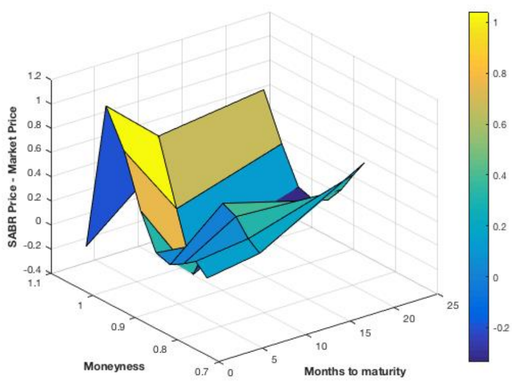
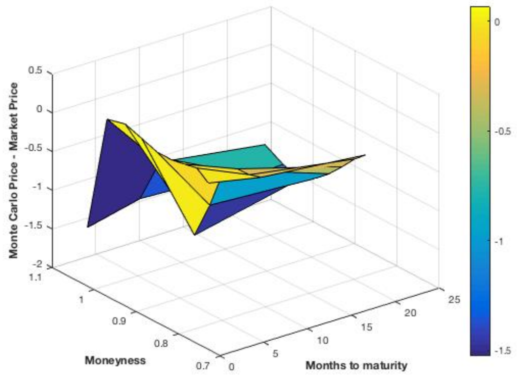


Figure 169: Zions Bancorporation Monte Carlo - Market price differential (\$) - Call - 11/04/16



When pricing the call (Figures 167 to 169), the Heston model prices it correctly when the option is out-of-the-money, and overestimates it otherwise by about \$0.3.

The SABR model portrays the same price differentials as the Heston model, except that it overprices the call by about \$0.2 when the option is out-of-the-money and by about \$0.7 when it is in-the-money.

The Monte Carlo methodology underprices the option at all strikes and maturities by about \$0.4, and by over \$1 when it has a few months to maturity and is far in-the-money.

Therefore, the Heston model is the winner for this particular case.

Let us now finally see how the three construction methodologies have performed with respect to the put option:

Table 48: Zions Bancorporation Mean of Implied Volatilities (%) - Put

	Jul/16	Jan/17	Jan/18
15	55.08	48.88	40.53
18	44.19	39.87	36.61
20	39.53	35.44	35.78
23	33.99	33.83	30.96
25	29.18	32.26	28.50
27	25.86	26.25	30.94

Table 49: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
Jul/16	0.15	0.5	-0.67	1.19
Jan/17	0.15	0.5	-0.62	0.78
Jan/18	0.14	0.5	-0.20	0.65

Figure 170: Zions Bancorporation Market Implied Volatility (%/100) - Put - 11/04/16

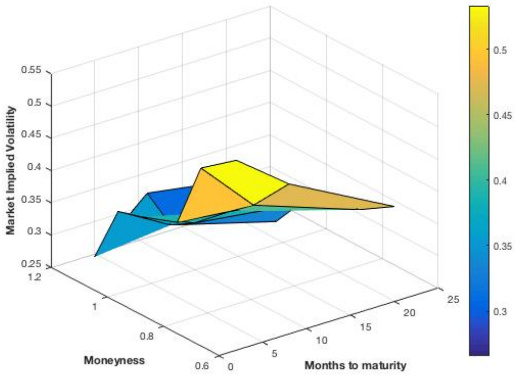


Figure 171: Zions Bancorporation Heston Implied Volatility (%/100) - Put - 11/04/16

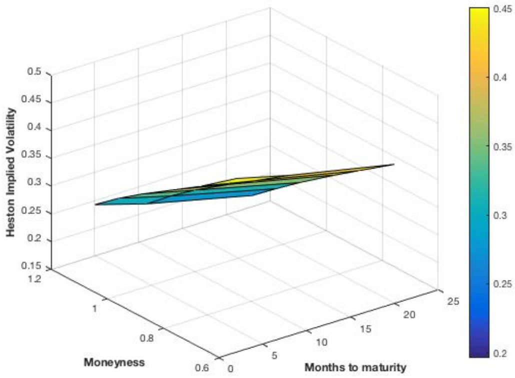


Figure 172: Zions Bancorporation SABR Implied Volatility (%/100) - Put - 11/04/16

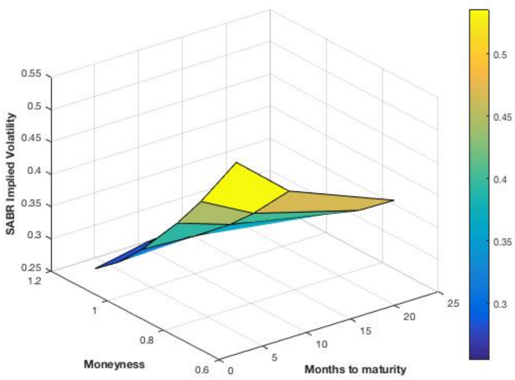
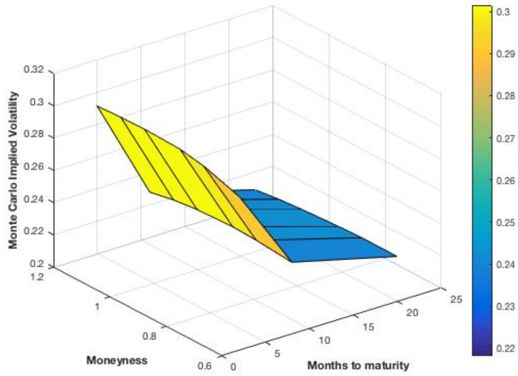


Figure 173: Zions Bancorporation Monte Carlo Implied Volatility (%/100) - Put - 11/04/16



In terms of estimating implied volatility (Figures 170 to 173), both the Heston and SABR model do a great job. Even though they to not manage to capture the various small bumps shown by the market implied volatility surface, they show the right average levels of implied volatilities for the various strikes and maturities.

The Monte Carlo model instead underestimates volatility greatly at all strikes and maturities, especially when the put has two years to maturity.

Figure 174: Zions Bancorporation Heston - Market price differential ($) - Put - 11/04/16

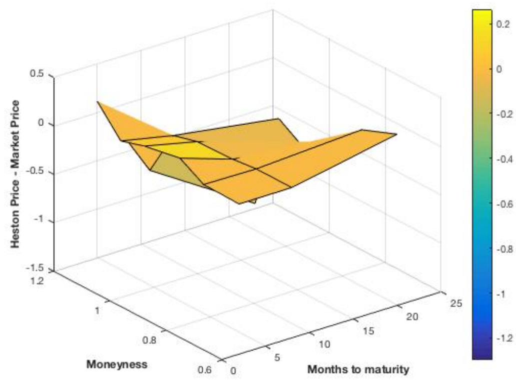


Figure 175: Zions Bancorporation SABR - Market price differential ($) - Put - 11/04/16

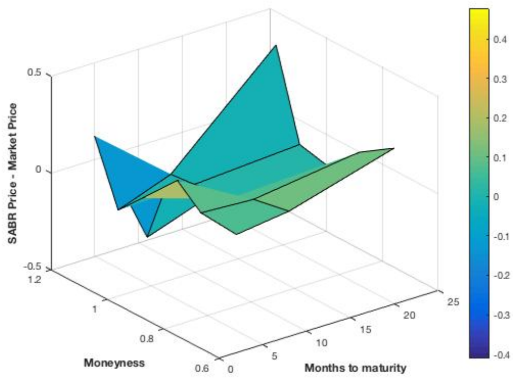
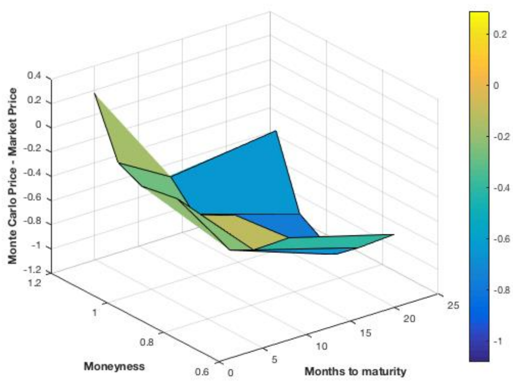


Figure 176: Zions Bancorporation Monte Carlo - Market price differential ($) - Put - 11/04/16



When comparing the three models with respect to the pricing of the put (Figures 174 to 176), we can see that the three are relatively more precise to price out-of-the-money puts, with mispricings almost $0 for the Heston and SABR models and not over $0.2 in absolute value for the Monte Carlo method.

When the put is in-the-money, the Heston model underprices it all the way down to $1.5 for puts with two years to maturity, whereas the SABR model overprices them up to $0.4 and the Monte Carlo method underprices them between an absolute value of $0.2 and $0.8.

Hence, the SABR construction methodology is the most precise in pricing the put for this particular instance.

8.5 Google

Google is a large-cap US technology firm ($504.9 billion[220]) with headquarters in Mountain View, California, and employs about 62000 people[221].

It has been chosen as a validate comparable company for this study because there is sufficient data available on the Internet concerning its option prices and implied volatilities (all of the following results are based on strikes, maturities, market prices and implied volatilities found at the following footnote[222]). Moreover, the way Google should be viewed different from Citigroup is exclusively in the sector it operates in (Information Technology vs. Financial Services).

A dividend of $0 per share[223] has been accounted for in the various calculations that follow

Let us now see the differences in implied volatilities between the three construction methodologies.

8.5.1 Underlying depreciation between 5% and 1%

In this specific instance the average implied volatilities used in the Heston and SABR models correspond to price decreases in the underlying of 1.005% and 1.65% occurred respectively on April 5^{th} 2016 and April 19^{th} 2016. The market data has been retrieved after a stock depreciation of 2.08% occurred on closing of business April 27^{th} 2016. Tables 50 to 53 show respectively the starting implied volatilities and SABR parameters used for this analysis.

Let us firstly see how the three models have performed with respect to the call option:

Table 50: Google Mean of Implied Volatilities (%) - Call

	May/16	Jun/16	Jan/17	Jan/18
700	31.11	29.48	27.15	28.58
710	38.02	30.25	29.62	28.23
720	29.47	26.40	27.83	28.14
730	29.12	25.82	27.91	27.08
740	28.46	25.44	26.88	28.10
750	27.98	24.88	25.91	27.47
760	27.34	24.36	25.45	26.82
770	28.13	24.02	25.86	27.22
780	26.44	23.51	24.98	26.08
790	25.91	23.18	26.56	25.84
800	25.40	22.91	24.79	26.39

Table 51: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
May/16	0.86	0.5	-0.66	2.40
Jun/16	0.75	0.5	-0.64	2.36
Jan/17	0.73	0.5	-0.48	1.03
Jan/18	0.73	0.5	-0.43	0.64

[220] http://finance.yahoo.com/q?s=GOOG
[221] http://finance.yahoo.com/q/pr?s=GOOG+Profile
[222] http://finance.yahoo.com/q/op?s=GOOG+Options
[223] http://finance.yahoo.com/q?s=GOOG

Figure 177: Google Market Implied Volatility (%/100) - Call - 26/04/16

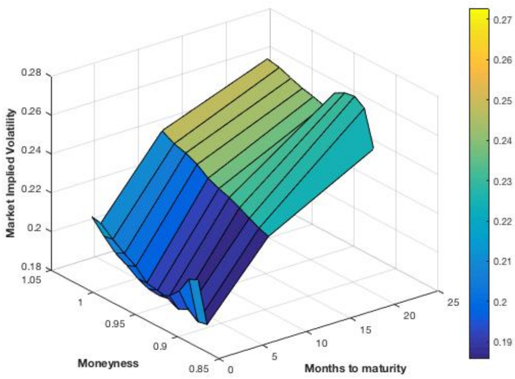


Figure 178: Google Heston Implied Volatility (%/100) - Call - 26/04/16

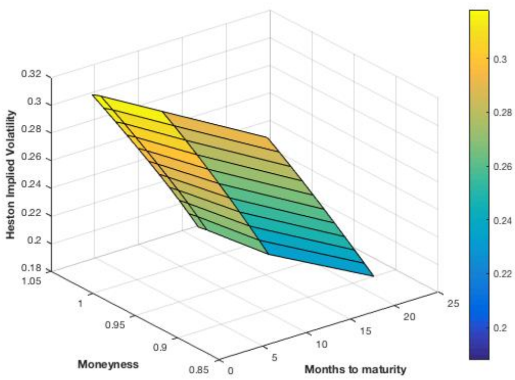


Figure 179: Google SABR Implied Volatility (%/100) - Call - 26/04/16

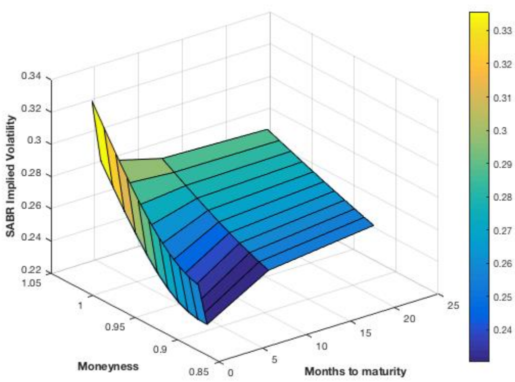
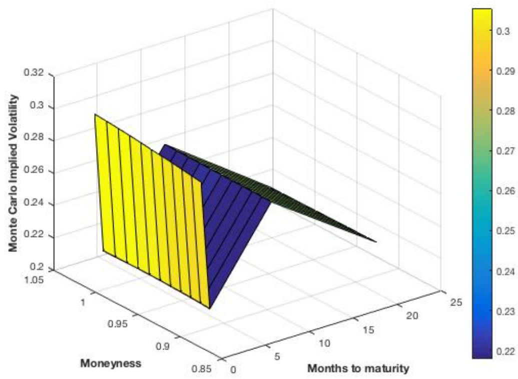


Figure 180: Google Monte Carlo Implied Volatility (%/100) - Call - 26/04/16



We can see that any of the three models is quite poor at approximating the implied volatility portrayed by the market (Figure 177). Hence, we will try to pick the less-worse model out of the three. Heston model-wise (Figure 178), it fails to portray the smiles at the various maturities and overestimates implied volatility for short-dated options. The same conclusions can be said for the SABR model (Figure 179). Both methodologies show skews instead of smiles.

The Monte Carlo process (Figure 180) does not show neither smiles nor skews and overestimates implied volatility for very short-dated options and does the opposite for longer-dated ones.

Figure 181: Google Heston - Market price differential ($) - Call - 26/04/16

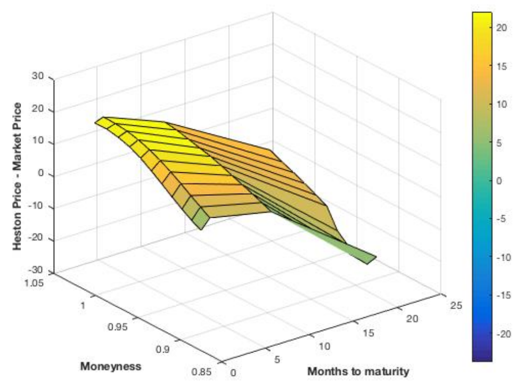


Figure 182: Google SABR - Market price differential ($) - Call - 26/04/16

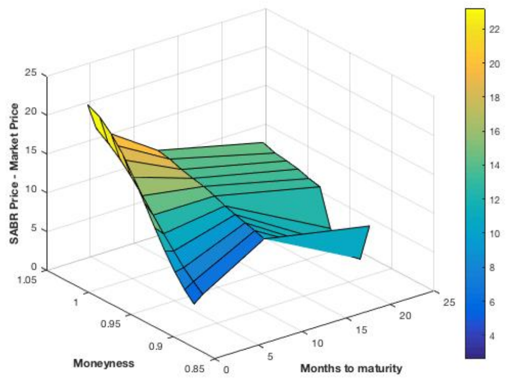
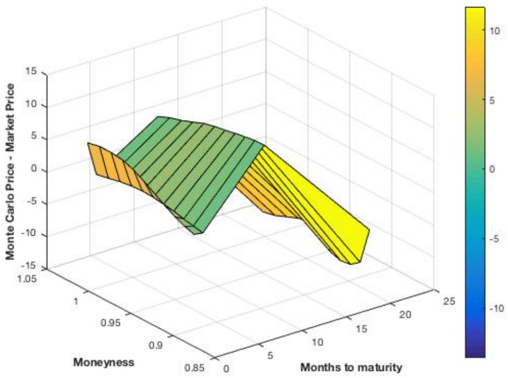


Figure 183: Google Monte Carlo - Market price differential ($) - Call - 26/04/16



When analysing the price differentials, we can see that the Heston model (Figure 181) overestimates the call price when the maturities are short to one year by about $15, while it underestimates the price of the options for a two-year maturity by about $10.

The SABR model (Figure 182) is slightly more stable since its differentials almost always overestimate the call price between $15 and $4.

But it is the Monte Carlo model (Figure 183) that is even more precise than the other two, having smaller price differentials (up to $5 of overestimations for short to medium maturities and $10 of underestimation for the option having two years to maturity).

Table 52: Google Mean of Implied Volatilities (%) - Put

	May/16	Jun/16	Jan/17	Jan/18
640	33.78	28.13	27.17	26.48
650	33.13	28.88	27.26	26.23
660	32.55	28.37	26.65	26.42
670	31.83	25.75	26.19	26.23
680	31.43	26.64	25.94	25.81
690	30.88	26.95	26.08	25.94
700	30.26	26.59	26.05	25.51
710	29.73	25.97	25.70	25.83
720	29.32	24.39	24.98	25.32
730	28.73	24.30	25.00	24.85
740	28.27	24.67	24.49	25.03

Table 53: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
May/16	0.79	0.5	-0.57	1.02
Jun/16	0.68	0.5	-0.82	0.56
Jan/17	0.68	0.5	-0.76	0.30
Jan/18	0.68	0.5	-0.58	0.14

Figure 184: Google Market Implied Volatility (%/100) - Put - 26/04/16

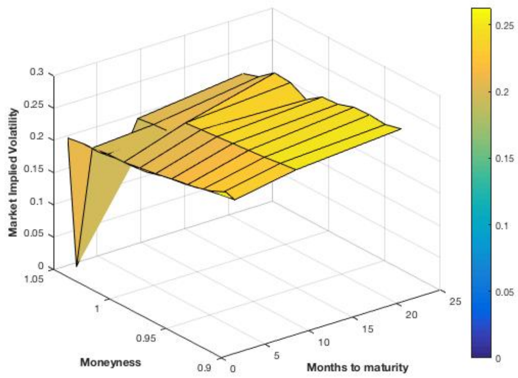


Figure 185: Google Heston Implied Volatility (%/100) - Put - 26/04/16

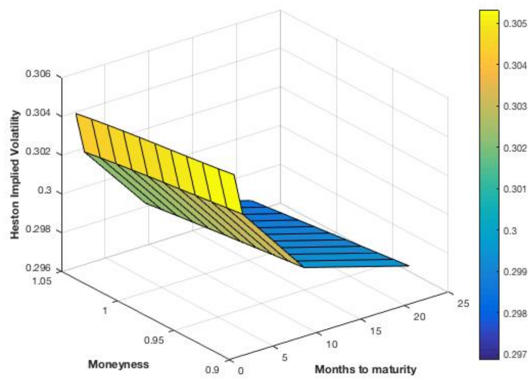


Figure 186: Google SABR Implied Volatility (%/100) - Put - 26/04/16

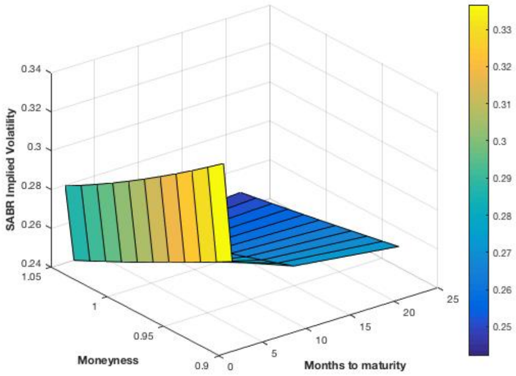
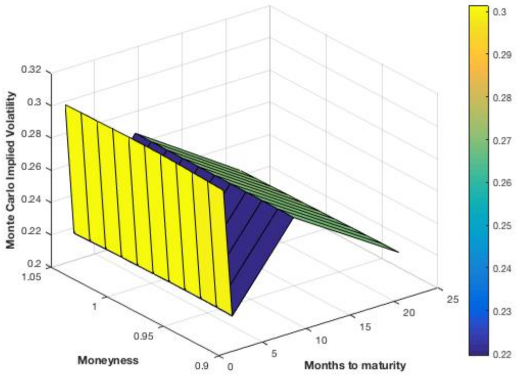


Figure 187: Google Monte Carlo Implied Volatility (%/100) - Put - 26/04/16



We can see that, except for an obvious underestimation of implied volatility by Yahoo! Finance, the market implied volatility for the put option (Figure 184) shows levels around the 20%-25% and presents neither skews nor smiles.

The Heston model (Figure 185) correctly shows the fact that there is no smile nor no skews, but it shows that implied volatility is decreasing as a function of time to maturity and overestimates it for all strikes and maturities.

The SABR model (Figure 186) also overestimates implied volatility at all strikes and times to maturity, but less so, and it shows a relatively flat surface for options having from a few months to maturity to two years to maturity.

The Monte Carlo model (Figure 187) correctly calibrates the implied volatilities of the option having a few months to maturity, and overestimates the other ones. It does not portray neither smiles nor skews, which is a good aspect of such analysis.

Figure 188: Google Heston - Market price differential ($) - Put - 26/04/16

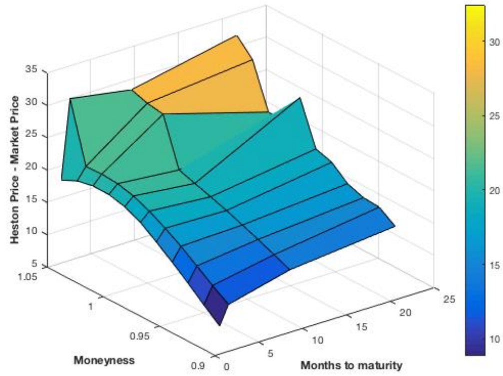


Figure 189: Google SABR - Market price differential ($) - Put - 26/04/16

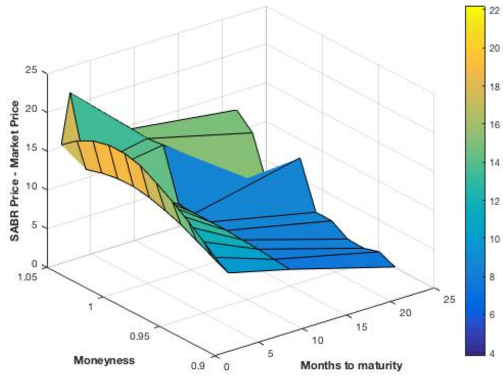
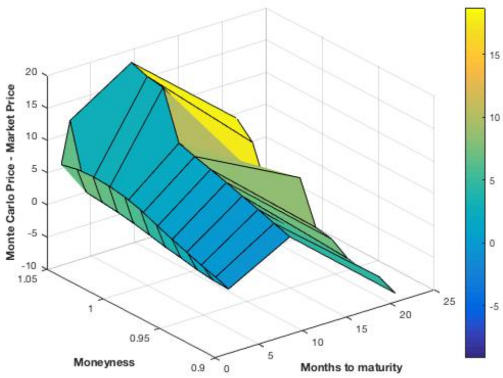


Figure 190: Google Monte Carlo - Market price differential ($) - Put - 26/04/16



In terms of price differentials, the Heston model (Figure 188) is more precise for the put being out-of-the-money, with differentials being around the $15 figure. As the option becomes more and more in-the-money, the overestimation increases up to \$30.

The SABR model (Figure 189) shows a similar pattern to that of the Heston model, although the overestimations are lower (around \$8 for out-of-the-money puts and of about \$15 for in-the-money puts).

The Monte Carlo process (Figure 190), instead, prices correctly the puts having a few months to maturity and slightly overestimates those with one month to maturity (about \$5 extra given to the prices) as well as those with one year to maturity (about \$5 of overestimation), while for the puts having two years to maturity the model underestimates prices by about \$5. Lastly, the model is more precise for out-of-the-money puts than for in-the-money ones.

So, we can say that the SABR and Monte Carlo models perform equally in this specific circumstance.

8.5.2 Underlying depreciation between 1% and 0%

For this study, the average implied volatilities for calls and puts have been retrieved after the stock has depreciated by respectively 0.30% and 0.74% on March 15^{th} 2016 and March 31^{st} respectively. The actual market data to which the Heston, SABR and Monte Carlo models will be compared to has been gotten after a stock depreciation of 0.62% occurred on April 4^{th} 2016.

Tables 54 to 57 show the average implied volatilities and SABR parameters respectively for calls and puts.

Table 54: Google Mean of Implied Volatilities (%) - Call

	May/16	Jun/16	Jan/17	Jan/18
700	28.90	27.15	27.35	28.13
710	28.06	26.27	27.16	29.92
720	27.48	25.85	26.84	27.73
730	27.02	25.31	26.62	27.10
740	26.46	24.81	26.36	27.41
750	26.03	24.43	26.11	27.21
760	25.54	24.01	25.86	26.66
770	25.29	23.60	25.85	26.85
780	24.83	23.20	25.62	26.68
790	24.26	22.86	25.42	26.44
800	24.07	22.52	25.13	26.44

Table 55: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
May/16	0.71	0.5	-0.48	1.25
Jun/16	0.66	0.5	-0.47	1.20
Jan/17	0.71	0.5	-0.33	0.60
Jan/18	0.70	0.5	-0.16	0.69

Figure 191: Google Market Implied Volatility (%/100) - Call - 04/04/16

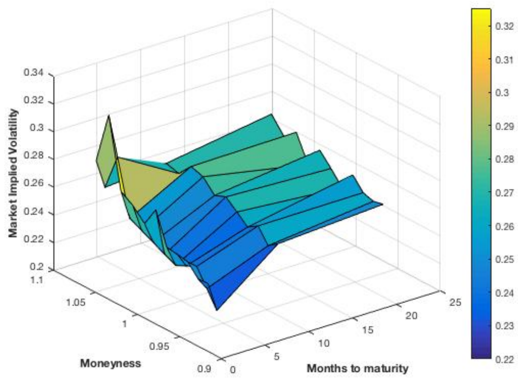


Figure 192: Google Heston Implied Volatility (%/100) - Call - 04/04/16

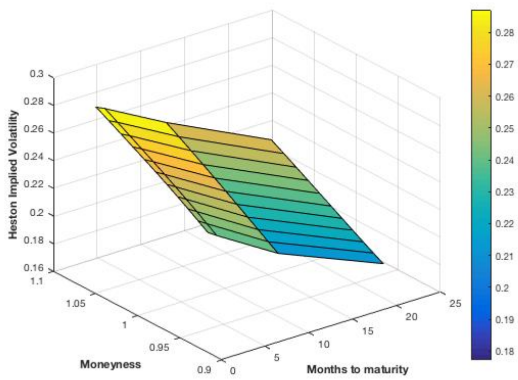


Figure 193: Google SABR Implied Volatility (%/100) - Call - 04/04/16

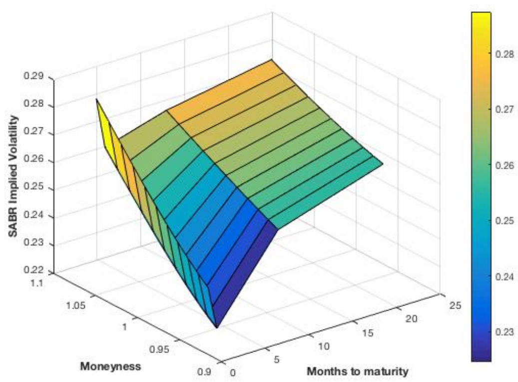
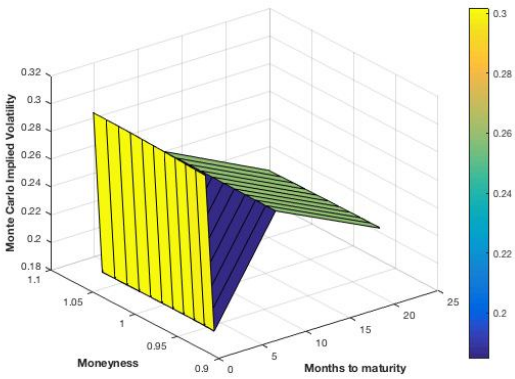


Figure 194: Google Monte Carlo Implied Volatility (%/100) - Call - 04/04/16



We can see that the market implied volatility surface (Figure 191) is quite particular, with implied volatility levels ranging form 22% to 32% and bumps in volatility for the same maturity. Thus, there is an absence of either smiles or skews.

However, the Heston (Figure 192) and SABR (Figure 193) models portray skews for all maturities. The SABR model also shows, correctly, that when the call has a few months to maturity implied volatility levels are lower than for other maturities.

The Monte Carlo process (Figure 194), instead, overestimates implied volatilities for the call having one month to maturity, and does the opposite for the call having a few months to maturity.

Figure 195: Google Heston - Market price differential ($) - Call - 04/04/16

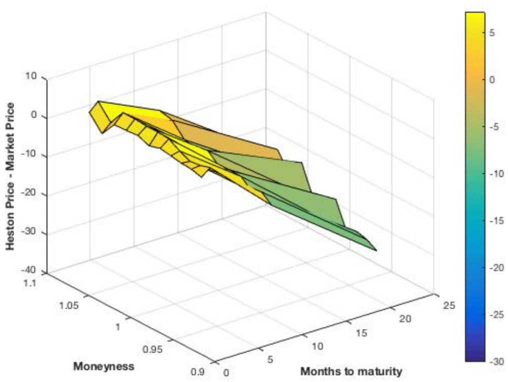


Figure 196: Google SABR - Market price differential ($) - Call - 04/04/16

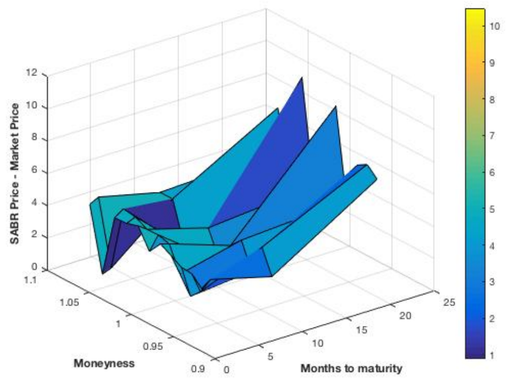
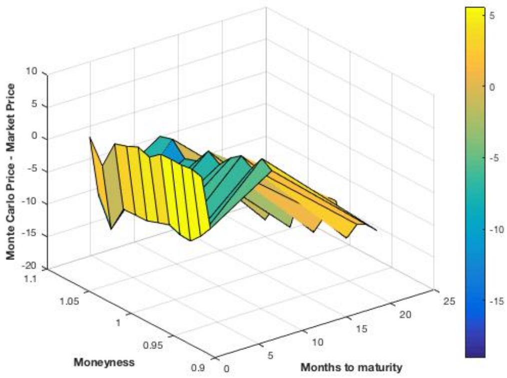


Figure 197: Google Monte Carlo - Market price differential ($) - Call - 04/04/16



And as well as in the previous analysis, the SABR model is the best one at approximating prices for the call option (Figure 196). Indeed, its mispricings are relatively low, not passing the $4 mark for the call having up to one year to maturity and not over the $10 mark for the call having two years to maturity.

While the Heston model (Figure 195) is quite precise for very short maturities, it underprices the call greatly for longer ones, arriving to pass the -$30 mark.

The Monte Carlo process (Figure 197) is also quite precise for very short maturities but it also underestimates the price of the call for longer ones, with the mispricings arriving to almost -$20.

Table 56: Google Mean of Implied Volatilities (%) - Put

	May/16	Jun/16	Jan/17	Jan/18
640	31.09	28.73	27.44	27.23
650	30.58	28.21	28.42	27.18
660	29.95	27.79	26.97	27.02
670	29.48	27.33	27.61	26.93
680	28.95	26.92	26.36	26.59
690	28.33	26.61	26.13	26.41
700	27.91	26.02	25.88	26.15
710	27.49	25.55	25.65	26.00
720	26.99	25.18	25.52	26.60
730	26.55	24.74	25.33	25.67
740	26.08	24.32	25.12	25.45

Table 57: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
May/16	0.70	0.5	-0.44	1.02
Jun/16	0.66	0.5	-0.51	0.82
Jan/17	0.65	0.5	-0.17	0.85
Jan/18	0.69	0.5	-0.24	0.34

Figure 198: Google Market Implied Volatility (%/100) - Put - 04/04/16

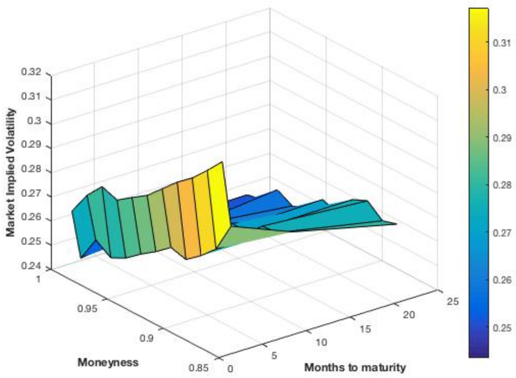


Figure 199: Google Heston Implied Volatility (%/100) - Put - 04/04/16

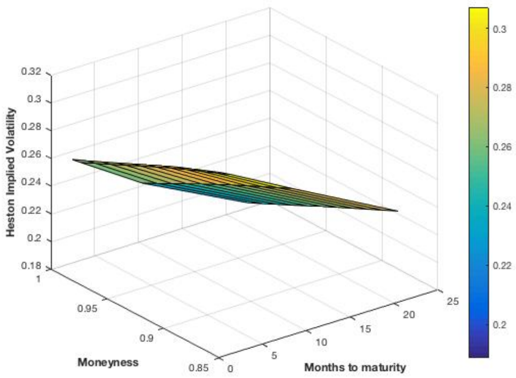


Figure 200: Google SABR Implied Volatility (%/100) - Put - 04/04/16

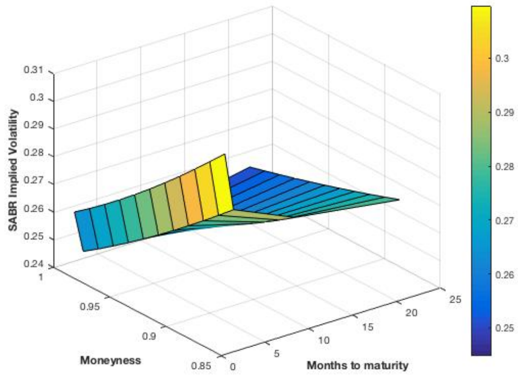
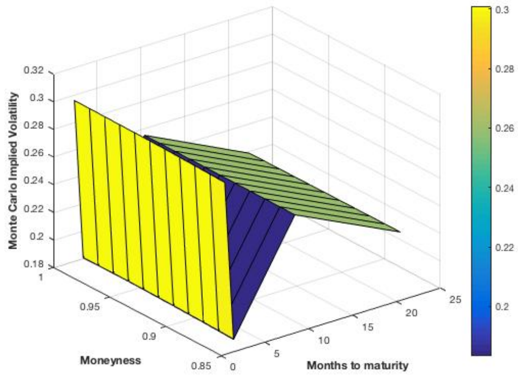


Figure 201: Google Monte Carlo Implied Volatility (%/100) - Put - 04/04/16



As we can see for the put option, the SABR model (Figure 200) is the best model to approximate the implied volatility shown by the market. Indeed, the higher volatility levels when the put has one month to maturity are captured perfectly, as well as the imperfect skews at the various maturities.

The Heston model (Figure 199) does a discrete job as well at underlining that implied volatility is highest for the shorter maturity. However, it fails to show the negative jump in implied volatility from the very first maturity to the second one.

The Monte Carlo process (Figure 201), once again, fails to portray the skews for the various maturities. However, it shows a negative jump in implied volatility from the one-month maturity put to the few-months maturity one, although the magnitude is bigger than what is actually shown in the market data.

Figure 202: Google Heston - Market price differential ($) - Put - 04/04/16

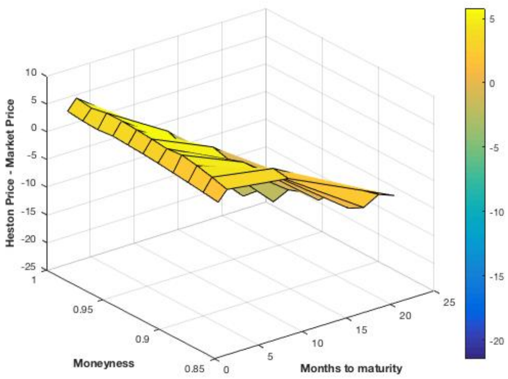


Figure 203: Google SABR - Market price differential ($) - Put - 04/04/16

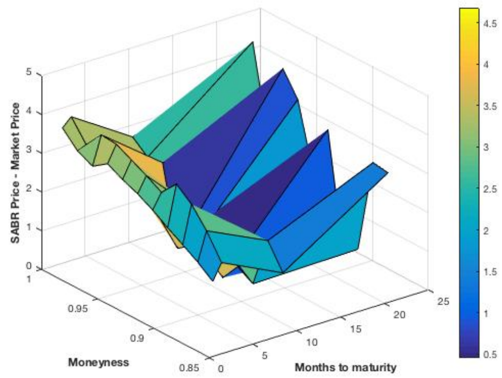
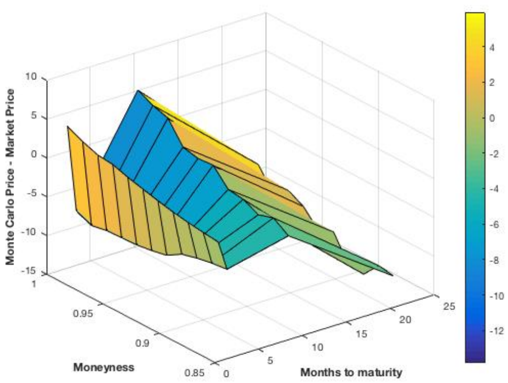


Figure 204: Google Monte Carlo - Market price differential ($) - Put - 04/04/16



Also in terms of price differentials the SABR model (Figure 203) is the one the most closely approaches the prices shown by the market. Indeed, the price differentials for such model do not go beyond \$4 for short maturities and \$5 for longer ones.

For the Heston and Monte Carlo models(Figures 202 and 204 respectively), instead, the differentials arrive to underpricings of the options of \$15 and \$10 respectively.

It must be underlined that the three models are quite precise to price the put with one month to maturity.

8.5.3 Underlying appreciation between 0% and 1%

In this instance the underlying appreciations are of 0.50% and 0.77% occurred on March 14^{th} 2016 and March 30^{th} 2016 respectively, and it has been through them that the implied volatilities averages have been taken. The actual market data, retrieved on April 1^{st} 2016, has been recorded as a stock appreciation of 0.67%. Tables 58 to 61 show the average implied volatilities and SABR calibrated parameters for calls and puts respectively.

Table 58: Google Mean of Implied Volatilities (%) - Call

	May/16	Jun/16	Jan/17	Jan/18
700	29.49	27.54	27.65	28.24
710	28.98	26.88	27.47	28.11
720	27.64	25.95	27.17	27.95
730	27.09	25.45	26.88	26.80
740	26.54	25.01	26.60	27.58
750	25.99	24.55	26.34	27.30
760	25.47	24.06	25.99	26.36
770	25.07	23.62	25.81	27.02
780	24.63	23.18	25.37	26.47
790	24.24	22.81	25.08	26.30
800	23.82	22.45	25.11	26.13

Table 59: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
May/16	0.70	0.5	-0.42	1.71
Jun/16	0.65	0.5	-0.46	1.37
Jan/17	0.71	0.5	-0.45	0.62
Jan/18	0.70	0.5	-0.23	0.73

Figure 205: Google Market Implied Volatility (%/100) - Call - 01/04/16

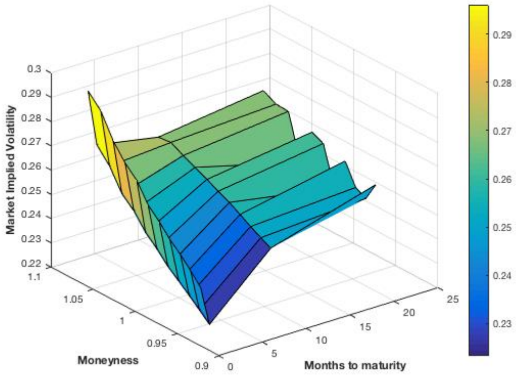


Figure 206: Google Heston Implied Volatility (%/100) - Call - 01/04/16

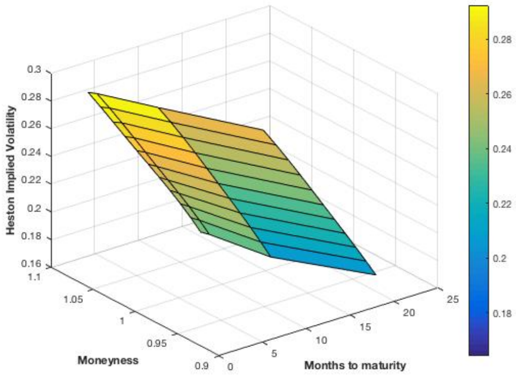


Figure 207: Google SABR Implied Volatility (%/100) - Call - 01/04/16

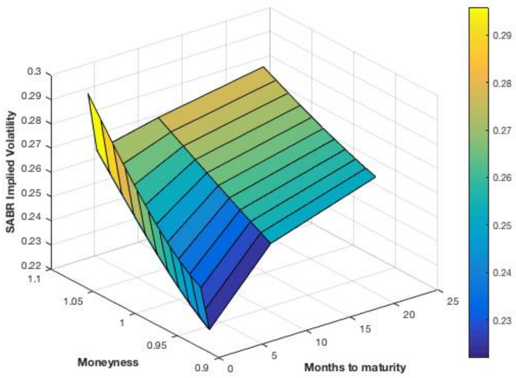
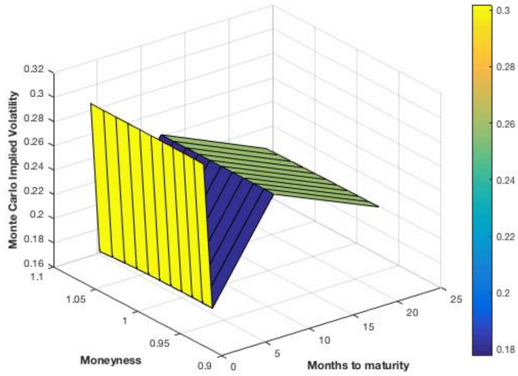


Figure 208: Google Monte Carlo Implied Volatility (%/100) - Call - 01/04/16



We can see that the SABR model (Figure 207) is the one that most closely matches the market implied volatility (Figure 205). It shows skews for all maturities and also shows the negative jump in implied volatilities form one-month-to-maturity puts to few-months-to-maturity ones. It also calibrates the levels of implied volatility at all levels.

Th Heston model (Figure 206) calibrates the implied volatilities slightly underestimating them at all levels, and failing to show the bump in implied volatility for middle maturities.

The Monte Carlo process (Figure 208) shows such negative jump, but the magnitude is bigger than what shown by the market. It also underestimates implied volatility for long maturities and fails to show the skews according to the various maturities.

Figure 209: Google Heston - Market price differential ($) - Call - 01/04/16

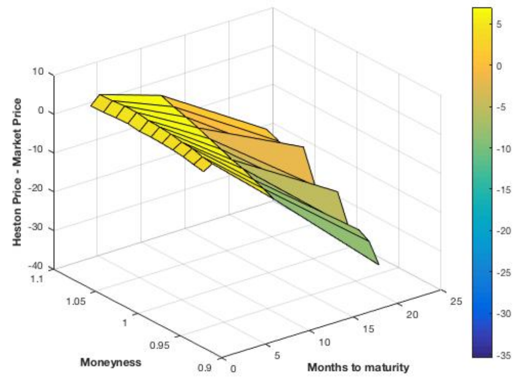


Figure 210: Google SABR - Market price differential ($) - Call - 01/04/16

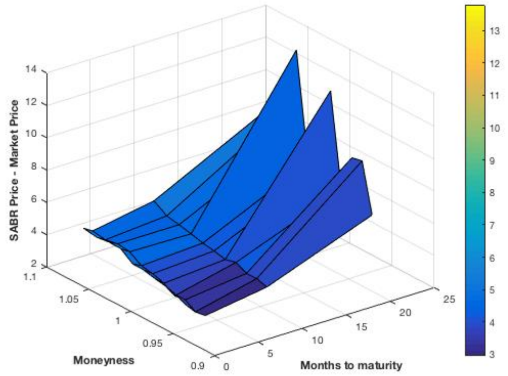
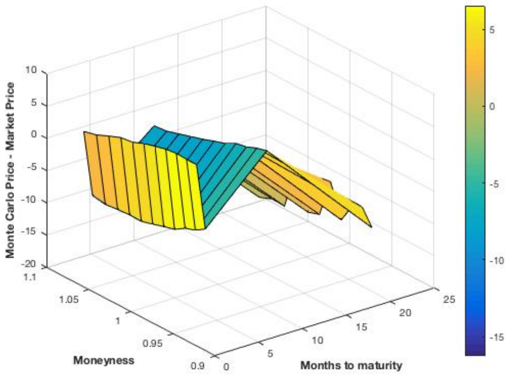


Figure 211: Google Monte Carlo - Market price differential ($) - Call - 01/04/16



Also in terms of pricing the call (Figures 209 to 211), the SABR model is the one that dominates. Indeed, while the three construction methodologies are quite precise for the call having form one month to a few months to maturity, for longer times to maturity the Heston model and Monte Carlo process underestimate the price of the call by respectively $30 and $15, while the SABR model overprices them by only up to $12.

Let us now see how the three models compare with respect to the put option:

Table 60: Google Mean of Implied Volatilities (%) - Put

	May/16	Jun/16	Jan/17	Jan/18
640	31.22	28.69	27.26	27.34
650	30.47	28.30	28.70	27.56
660	29.65	27.76	28.22	27.41
670	29.26	27.25	27.83	27.18
680	28.96	26.60	26.13	26.46
690	28.53	26.08	25.86	26.31
700	27.77	25.77	26.06	26.20
710	27.35	25.43	25.83	25.93
720	26.70	25.06	25.56	26.92
730	26.34	24.55	25.16	25.62
740	25.94	24.05	25.00	25.85

Table 61: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
May/16	0.69	0.5	-0.43	1.06
Jun/16	0.64	0.5	-0.38	0.99
Jan/17	0.66	0.5	-0.40	0.69
Jan/18	0.66	0.5	0.00	0.70

Figure 212: Google Market Implied Volatility (%/100) - Put - 01/04/16

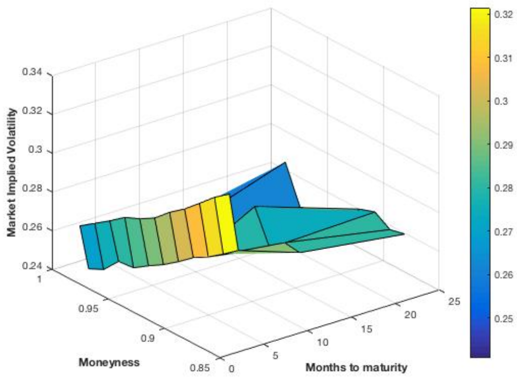


Figure 213: Google Heston Implied Volatility (%/100) - Put - 01/04/16

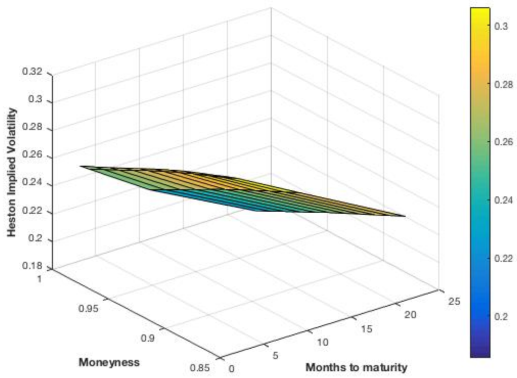


Figure 214: Google SABR Implied Volatility (%/100) - Put - 01/04/16

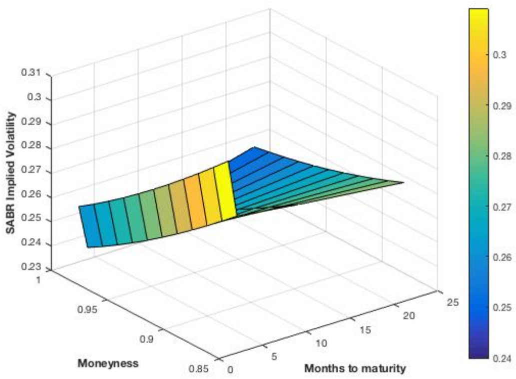
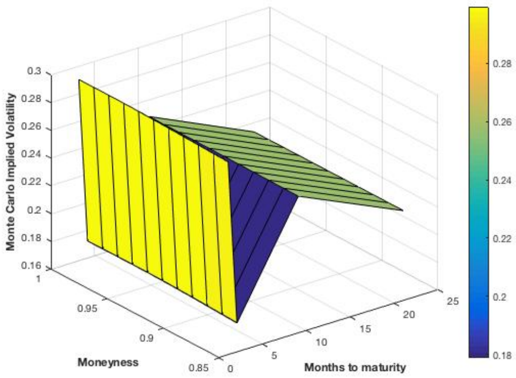


Figure 215: Google Monte Carlo Implied Volatility (%/100) - Put - 01/04/16



Once again, the SABR model (Figure 214) is the only one that correctly calibrates implied volatility levels at all strikes and maturities, managing to show the skews for the various maturities and the spike in volatility for the put with one month to maturity.

The Heston model (Figure 213) does a discrete job as well, but it fails to show the spike in volatility for the one-month-to-maturity put and slightly underestimates the implied volatility levels for longer maturities.

The Monte Carlo methodology (Figure 215) overestimates implied volatility for the put having one month to maturity and, once again, does not portray the skews for the various maturities.

Figure 216: Google Heston - Market price differential ($) - Put - 01/04/16

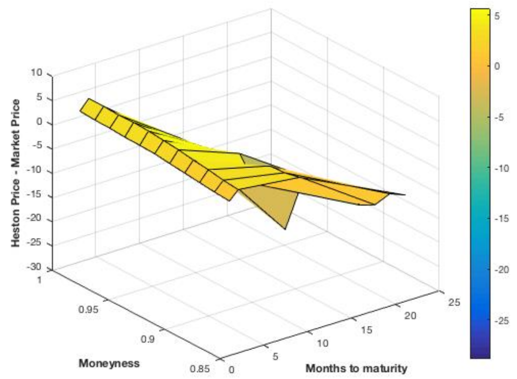


Figure 217: Google SABR - Market price differential ($) - Put - 01/04/16

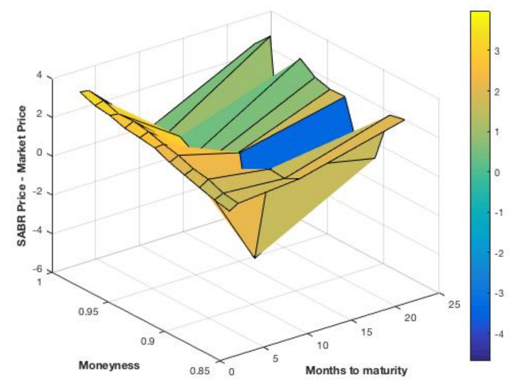
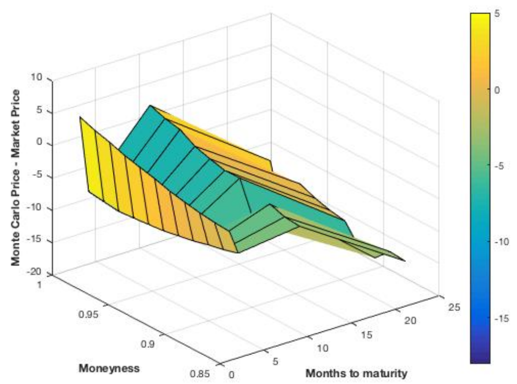


Figure 218: Google Monte Carlo - Market price differential ($) - Put - 01/04/16



For the pricing of the put (Figures 216 to 218), the SABR model matches prices the closest to the market data. Indeed, the mispricings only arrive up to -$4 for middle maturities, while the Heston model and the Monte Carlo process have mispricings that reach over the -$15 figure for very long maturities.

It must be underlined that the three models are quite precise for the put having one month to maturity.

8.5.4 Underlying appreciation between 1% and 5%

Unfortunately, between March 2016 and April 2016, which was the period during which the empirical experimentation has been carried out, the stock of Google has moved above 1% only twice, not guaranteeing the full spectrum of analysis of implied volatilities and option price calibrations. Thus, there will be no further examination for this specific instance.

8.6 Exxon Mobil

Exxon Mobil is a large market cap ($344 billion[224]) US enterprise. It is involved in the Oil & Gas business through the production of crude Oil & Natural Gas globally. Headquartered in Irving, Texas, it employs 73500 people[225].

Exxon Mobil has been chosen as a validate candidate for this study because of two major reasons: its vastly greater market cap than that of Citigroup and because of the difference in sector specialisation of the two companies. Therefore, the way Exxon Mobil should be viewed differently from Citigroup is in the two characteristics listed previously. All of the information on strikes, maturities, market prices and implied volatilities can be found at the following footnote[226].

A dividend of $2.92 per share[227] has been accounted for in the calculations of implied volatilities and option prices.

8.6.1 Underlying depreciation between 5% and 1%

For this study, the averaged implied volatilities have been extracted from market data recorded after stock depreciations of 1.10% and 1.14% recorded respectively on March 31^{st} 2016 and April 5^{th} 2016. The actual market data used to compare to the three models has been retrieved after a stock depreciation of 1.13% occurred on April 7^{th} 2016.

Tables 62 to 65 show the average implied volatilities and SABR calibrated parameters that have been used to estimate both implied volatilities and option prices.

[224] http://finance.yahoo.com/q?s=XOM
[225] http://finance.yahoo.com/q/pr?s=XOM+Profile
[226] http://finance.yahoo.com/q/op?s=XOM+Options
[227] http://finance.yahoo.com/q?s=XOM

Table 62: Exxon Mobil Mean of Implied Volatilities (%) - Call

	May/16	Jul/16	Jan/17	Jan/18
75	28.75	21.41	20.32	18.55
77.5	23.63	22.34	19.59	19.06
80	21.48	18.97	18.90	18.30
82.5	18.97	17.83	18.34	17.46
85	17.45	16.83	17.77	17.05
87.5	16.38	16.09	17.10	16.75
90	15.87	15.62	16.54	16.17
92.5	16.70	15.35	16.52	16.11
95	17.73	15.44	16.25	16.03

Table 63: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
May/16	0.17	0.5	-0.62	2.44
Jul/16	0.16	0.5	-0.56	1.27
Jan/17	0.16	0.5	-0.48	0.65
Jan/18	0.16	0.5	-0.52	0.37

Figure 219: Exxon Mobil Market Implied Volatility (%/100) - Call - 07/04/16

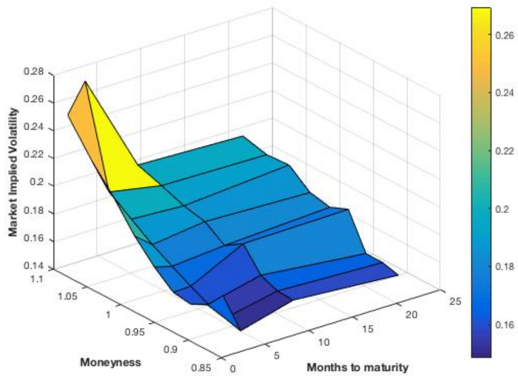


Figure 220: Exxon Mobil Heston Implied Volatility (%/100) - Call - 07/04/16

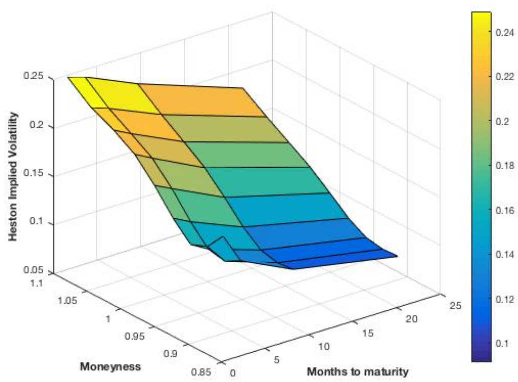


Figure 221: Exxon Mobil SABR Implied Volatility (%/100) - Call - 07/04/16

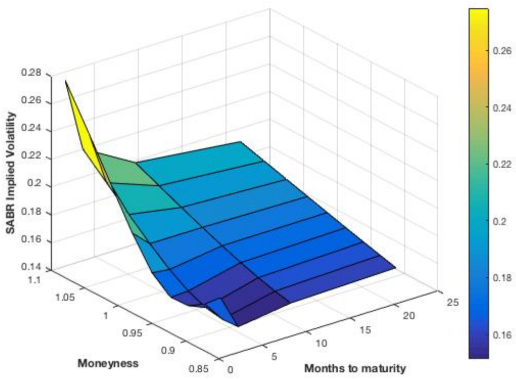
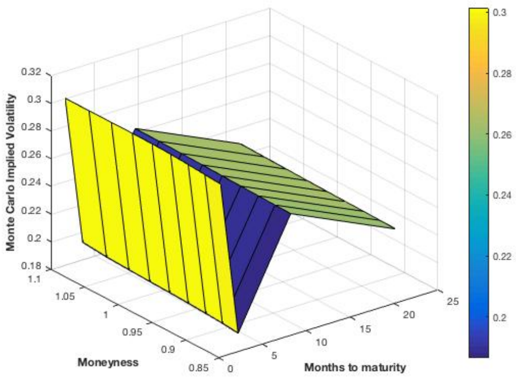


Figure 222: Exxon Mobil Monte Carlo Implied Volatility - Call - 07/04/16



We can see, when analysing Figures 219 to 222, that the SABR model is the one that best approaches implied volatility to that shown by the market. Indeed, it correctly shows skews for all maturities and the spike in volatility when the call is in-the-money with a few months to maturity.

The same can be said for the Heston model, except for the missing out of the portrayal in spike in volatility when the call is in-the-money with a few months to maturity.

The Monte Carlo model overestimates the implied volatility of the call when it has one month to maturity, does the opposite when it has 2 months to maturity and fails to show the skews for the various maturities, as it has often been the case for other instances.

Figure 223: Exxon Mobil Heston - Market price differential ($) - Call - 07/04/16

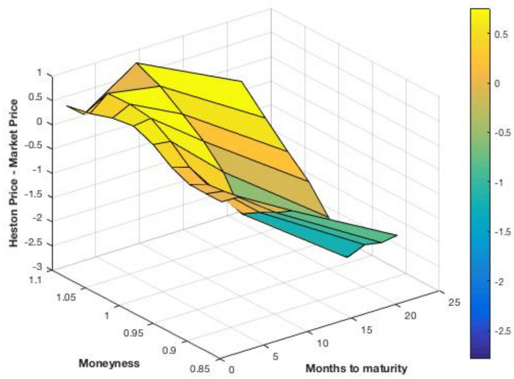


Figure 224: Exxon Mobil SABR - Market price differential ($) - Call - 07/04/16

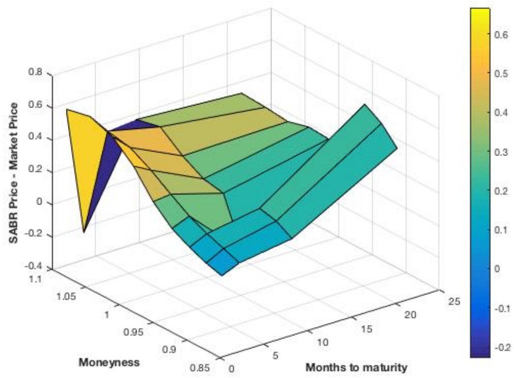
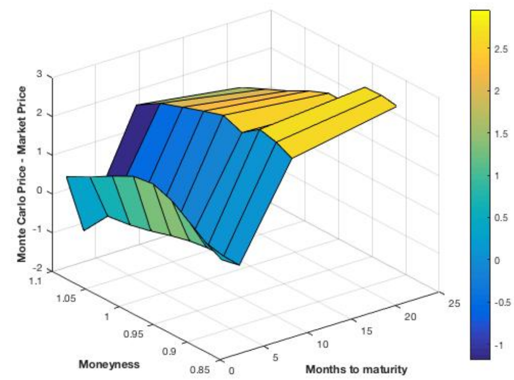


Figure 225: Exxon Mobil Monte Carlo - Market price differential ($) - Call - 07/04/16



In terms of pricing of the call (Figures 223 to 225), the SABR model is also here the best model to use. Indeed, the discrepancies in prices do not go over the absolute value of \$0.8, while they reach \$2 for the other two models.

The SABR model is especially precise when the call is out-of-the-money, as well as the Monte Carlo method.

The Heston model, instead, is more precise when the call is in-the-money.

Let us now observe how the three models have performed with respect of the put:

Table 64: Exxon Mobil Mean of Implied Volatilities (%) - Put

	May/16	Jul/16	Jan/17	Jan/18
70	31.18	26.88	27.45	25.89
72.5	29.14	26.30	26.49	25.39
75	27.31	24.83	25.57	24.25
77.5	25.67	23.45	24.73	24.18
80	24.15	22.21	23.93	23.88
82.5	23.02	21.10	23.38	22.80
85	22.77	20.43	23.04	22.16

Table 65: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
May/16	0.20	0.5	-0.21	1.87
Jul/16	0.19	0.5	-0.63	0.90
Jan/17	0.20	0.5	-0.22	0.95
Jan/18	0.21	0.5	-0.79	0.33

Figure 226: Exxon Mobil Market Implied Volatility (%/100) - Put - 07/04/16

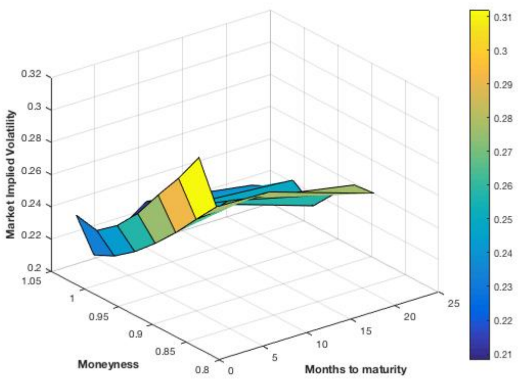


Figure 227: Exxon Mobil Heston Implied Volatility (%/100) - Put - 07/04/16

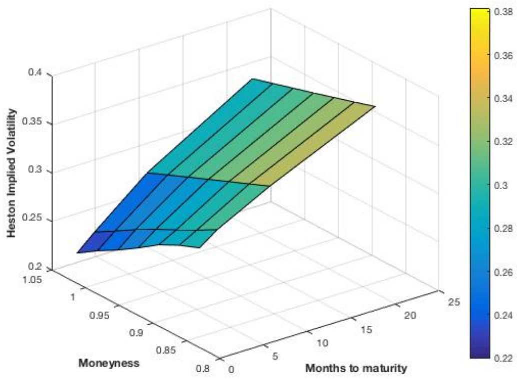


Figure 228: Exxon Mobil SABR Implied Volatility (%/100) - Put - 07/04/16

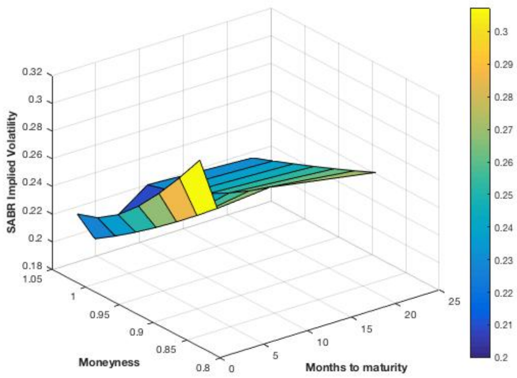
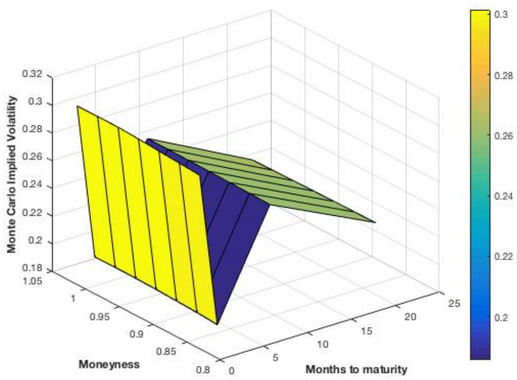


Figure 229: Exxon Mobil Monte Carlo Implied Volatility (%/100) - Put - 07/04/16



As the Figures 226 to 229 show, the SABR model is by far the closest one in terms of portraying implied volatility that approaches the market data. It shows the smile when the put has one month to maturity and the skews for other maturities. Moreover, the values of implied volatility are calibrated correctly.

The Heston model, instead, overestimates the implied volatilities of the put for long maturities and does not show the smile when the option has one month to maturity.

The Monte Carlo model fails to show the skews and smiles respectively for long and short maturities, overestimates volatility when the put has one one month to maturity, and does the opposite when the option has two months to maturity.

Figure 230: Exxon Mobil Heston - Market price differential ($) - Put - 07/04/16

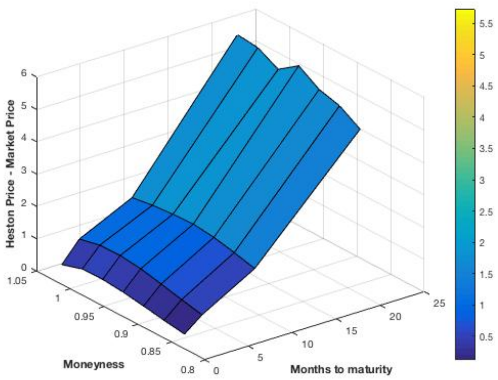


Figure 231: Exxon Mobil SABR - Market price differential ($) - Put - 07/04/16

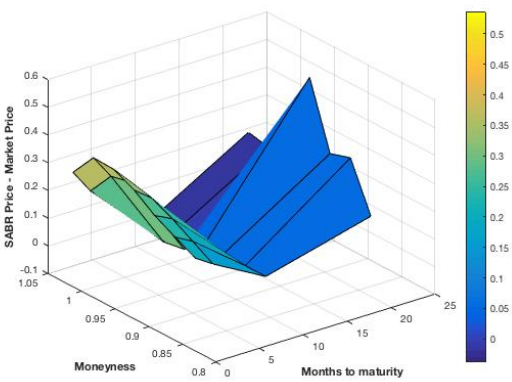
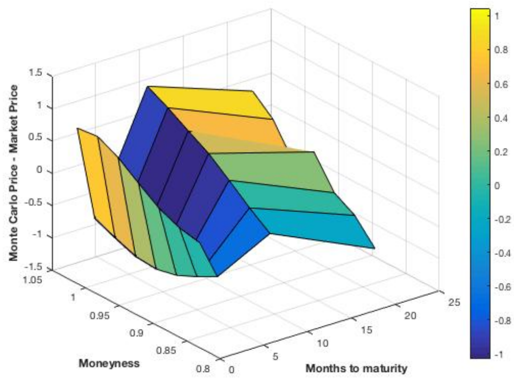


Figure 232: Exxon Mobil Monte Carlo - Market price differential ($) - Put - 07/04/16



Pricing-wise (Figures 230 to 232) the SABR model dominates the other two. Indeed, the biggest differentials are of $0.4 when the put is at-the-money with two years to maturity.

For the Heston model, as the option becomes more long-dated, the discrepancy in prices augments, reaching almost $6.

For the Monte Carlo model there is an underpricing of the put of $1 when the option has 2 months to maturity, while for the rest of maturities it is quite precise.

8.6.2 Underlying depreciation between 1% and 0%

The implied volatility averages have been taken after two stocks depreciations of respectively 0.69% (on March 21^{st} 2016) and 0.01% (on March 30^{th} 2016). The actual market implied volatilities and prices have been retrieved after a stock depreciation of 0.75% occurred on April 1^{st} 2016.

Tables 66 to 69 show the mean implied volatilities and calibrated SABR parameters used to estimate the new implied volatilities and option prices by the three models.

Table 66: Exxon Mobil Mean of Implied Volatilities (%) - Call

	May/16	Jul/16	Jan/17	Jan/18
75	20.58	21.38	19.24	18.28
77.5	22.78	19.53	18.50	17.90
80	20.83	18.73	17.87	17.78
82.5	18.87	17.34	17.23	16.82
85	17.04	16.17	16.64	16.28
87.5	15.85	15.36	16.01	16.28
90	15.20	14.71	15.43	15.73
92.5	14.82	14.19	15.00	15.62
95	15.70	14.13	14.85	15.54

Table 67: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
May/16	0.17	0.5	-0.60	1.15
Jul/16	0.15	0.5	-0.59	1.11
Jan/17	0.15	0.5	-0.56	0.58
Jan/18	0.15	0.5	-0.36	0.52

Figure 233: Exxon Mobil Market Implied Volatility (%/100) - Call - 01/04/16

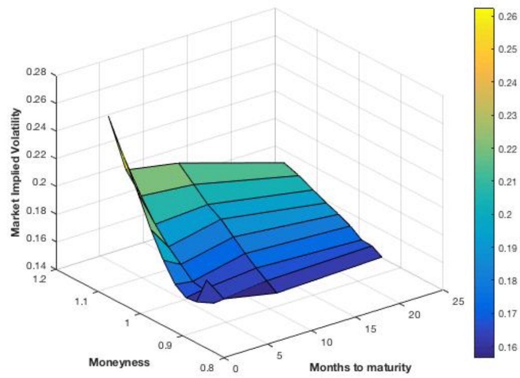


Figure 234: Exxon Mobil Heston Implied Volatility (%/100) - Call - 01/04/16

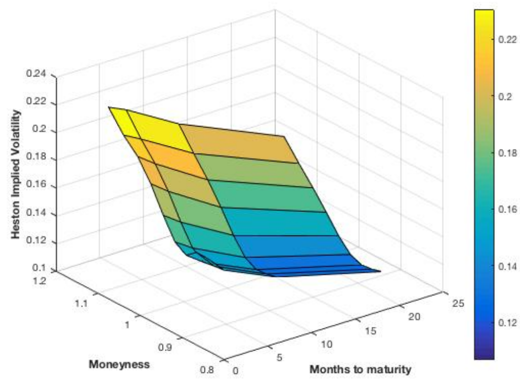


Figure 235: Exxon Mobil SABR Implied Volatility (%/100) - Call - 01/04/16

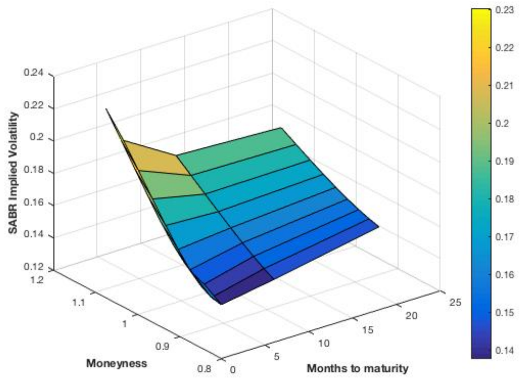
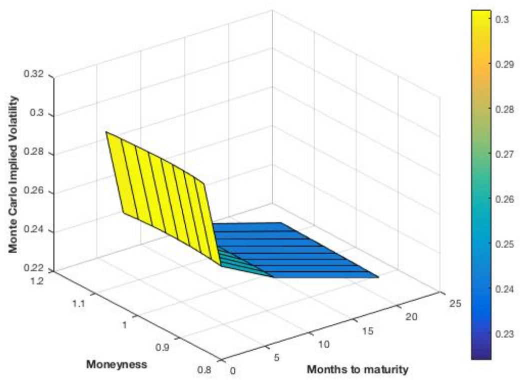


Figure 236: Exxon Mobil Monte Carlo Implied Volatility (%/100) - Call - 01/04/16



As we can see from Figures 233 to 236, both the Heston and SABR models do a discrete job in portraying the implied volatility when compared to the market data.

The SABR model shows the spike in volatility when the call is in-the-money with one month to maturity but does not show the slight smile for the same maturity, while vice-versa can be said for the Heston model. Thus, both models calibrate the levels of implied volatilities for almost all strikes and maturities.

The Monte Carlo method instead overestimates implied volatility for all strikes and maturities and fails to show the smiles and skews respectively for short dates and longer ones.

Figure 237: Exxon Mobil Heston - Market price differential ($) - Call - 01/04/16

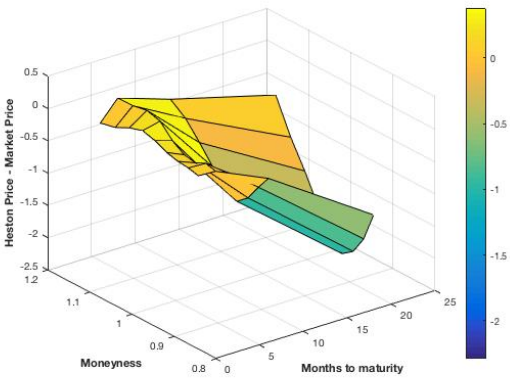


Figure 238: Exxon Mobil SABR - Market price differential ($) - Call - 01/04/16

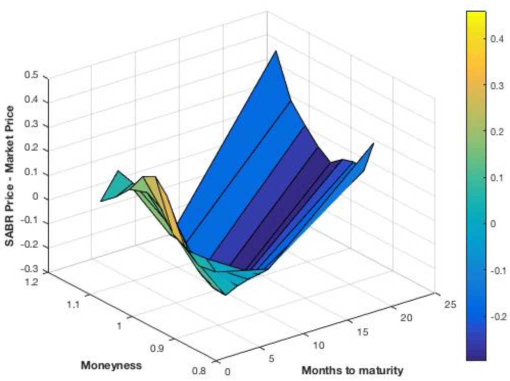
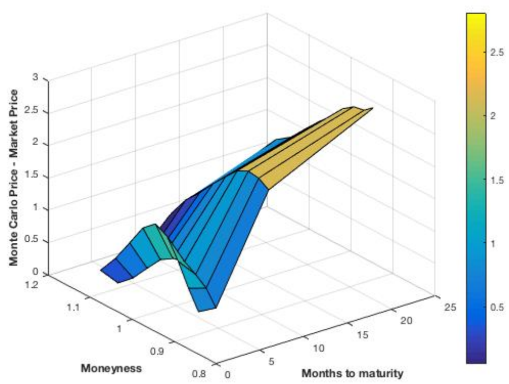


Figure 239: Exxon Mobil Monte Carlo - Market price differential ($) - Call - 01/04/16



However, in terms of price differential (Figures 237 to 239), the SABR model is superior to the other two. Indeed, the discrepancies in prices do not go over an absolute value of $0.4, while for the Heston model the discrepancies reach the -$1.5 figure and for the Monte Carlo process the overpricing reaches the $2 figure.

Lastly, it can be noticed that the three models are quite precise at pricing the call when it has one month to maturity.

Table 68: Exxon Mobil Mean of Implied Volatilities (%) - Put

	May/16	Jul/16	Jan/17	Jan/18
70	30.35	27.61	26.73	25.50
72.5	28.55	25.97	25.84	24.91
75	26.76	24.33	24.94	24.08
77.5	24.92	23.03	24.06	23.75
80	23.28	21.72	23.15	23.50
82.5	21.99	20.51	22.51	22.94
85	21.36	19.85	21.83	22.62

Table 69: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
May/16	0.19	0.5	-0.31	1.70
Jul/16	0.18	0.5	-0.34	1.38
Jan/17	0.20	0.5	-0.47	0.72
Jan/18	0.21	0.5	-0.66	0.26

Figure 240: Exxon Mobil Market Implied Volatility (%/100) - Put - 01/04/16

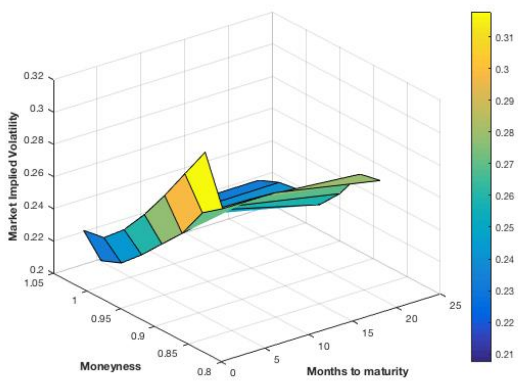


Figure 241: Exxon Mobil Heston Implied Volatility (%/100) - Put - 01/04/16

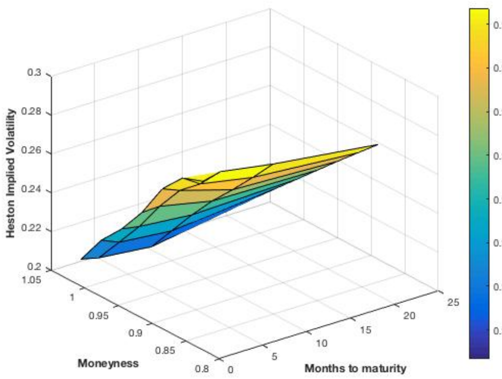


Figure 242: Exxon Mobil SABR Implied Volatility (%/100) - Put - 01/04/16

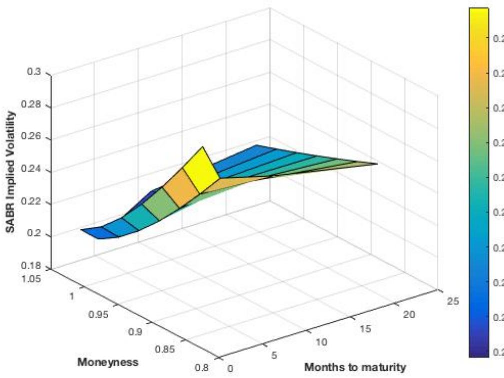
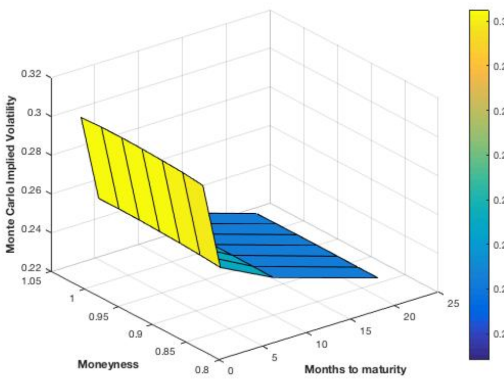


Figure 243: Exxon Mobil Monte Carlo Implied Volatility (%/100) - Put - 01/04/16



Once again, the shape of the market implied volatility surface (Figure 240) is best approximated by the SABR model (Figure 242). It calibrates volatilities correctly at all strikes and maturities and shows the skews correctly for all maturities.

The Heston model (Figure 241) does a great job as well, but fails to show that actual implied volatility is far higher for the put having one month to maturity than for other maturities.

The Monte Carlo process (Figure 243), while it is precise for the put having two years to maturity, it overestimates implied volatility for shorter maturities and fails to show the skews for each single maturity.

Figure 244: Exxon Mobil Heston - Market price differential ($) - Put - 01/04/16

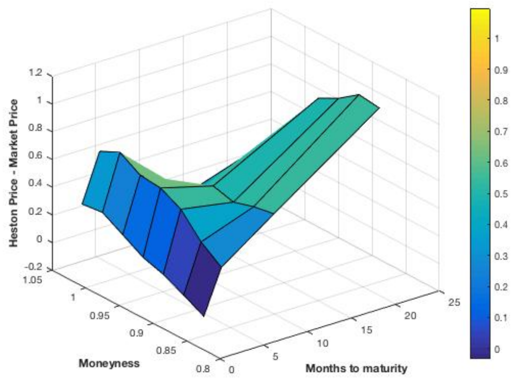


Figure 245: Exxon Mobil SABR - Market price differential ($) - Put - 01/04/16

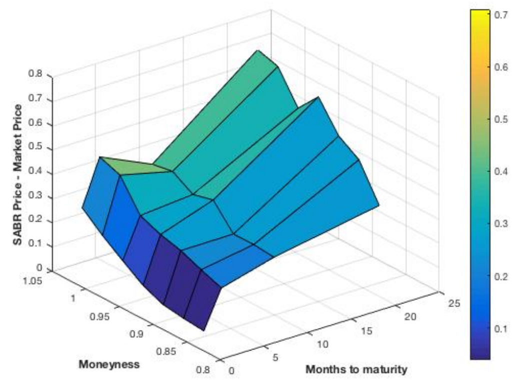
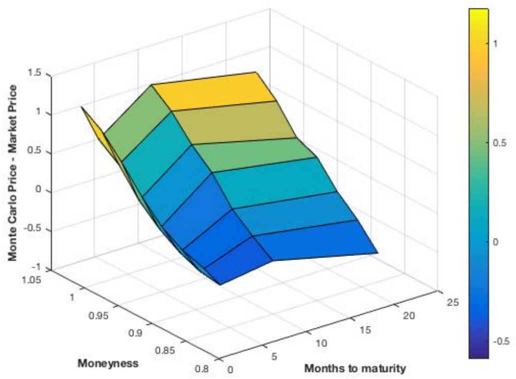


Figure 246: Exxon Mobil Monte Carlo - Market price differential ($) - Put - 01/04/16



In terms of pricing the put (Figures 244 to 246), the SABR model is the most precise. Indeed, the discrepancies do not go beyond the $0.7 figure, while for the Heston model they arrive to $1 overpricing and for Monte Carlo they pricing difference ranges form -$1 to +$1.

It is interesting to notice that both the SABR and Heston model lose precision as the put has a greater time to maturity.

8.6.3 Underlying appreciation between 0% and 1%

In this case the two stock appreciations that have been taken into account to find the average implied volatilities to use for the SABR and Heston model are the following: 0.50% on March 15^{th} 2016 and 0.24% occurred on April 4^{th} 2016. The actual market data has been retrieved after a stock appreciation of 0.13% that happened on April 11^{th} 2016. The mean of implied volatilities with the respective SABR calibrated parameters can be seen in Tables 70 to 73.

Table 70: Exxon Mobil Mean of Implied Volatilities (%) - Call

	May/16	Jul/16	Jan/17	Jan/18
75	24.50	21.50	20.76	19.02
77.5	22.69	20.54	19.92	19.40
80	20.91	19.03	19.21	18.69
82.5	19.17	18.08	18.33	18.20
85	17.81	17.04	17.62	17.73
87.5	16.65	16.25	17.32	17.56
90	15.81	15.62	16.97	17.01
92.5	16.36	15.22	16.58	16.77
95	17.19	15.74	16.23	16.50

Table 71: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
May/16	0.17	0.5	-0.51	1.75
Jul/16	0.16	0.5	-0.49	1.16
Jan/17	0.16	0.5	-0.43	0.76
Jan/18	0.17	0.5	-0.80	0.20

Figure 247: Exxon Mobil Market Implied Volatility (%/100) - Call - 11/04/16

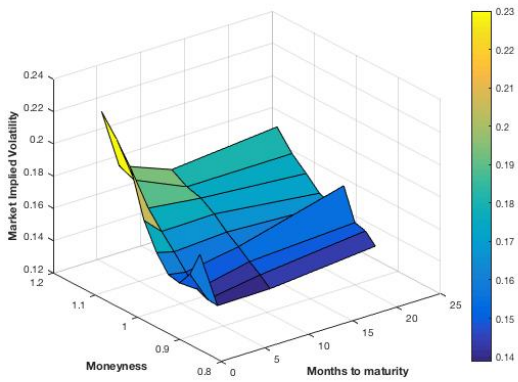


Figure 248: Exxon Mobil Heston Implied Volatility (%/100) - Call - 11/04/16

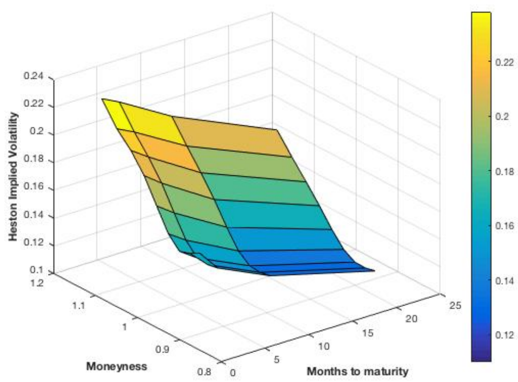


Figure 249: Exxon Mobil SABR Implied Volatility (%/100) - Call - 11/04/16

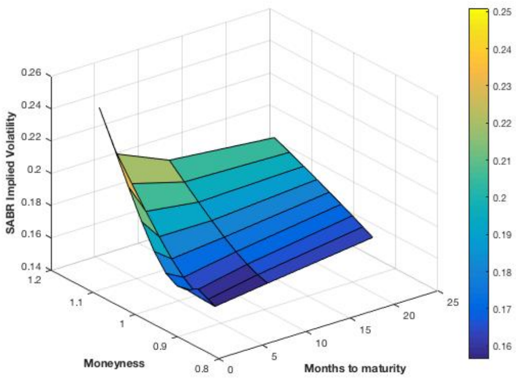
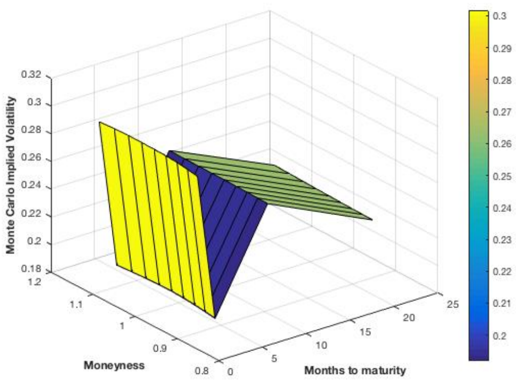


Figure 250: Exxon Mobil Monte Carlo Implied Volatility (%/100) - Call - 11/04/16



In terms of portraying implied volatility for the call (Figures 247 to 250), both the Heston and SABR model perform quite well, calibrating the implied volatilities correctly for almost all maturities and strikes and showing the smile for the one month to maturity call and skews for the other maturities.

The Monte Carlo methodology, on the other hand, overestimates implied volatility at all maturities and strikes, failing additionally to show the smiles and skews according to time to maturity.

Figure 251: Exxon Mobil Heston - Market price differential ($) - Call - 11/04/16

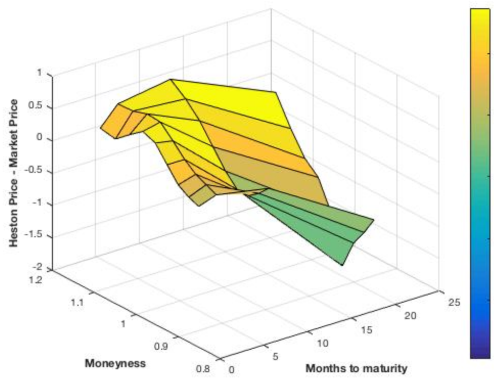


Figure 252: Exxon Mobil SABR - Market price differential ($) - Call - 11/04/16

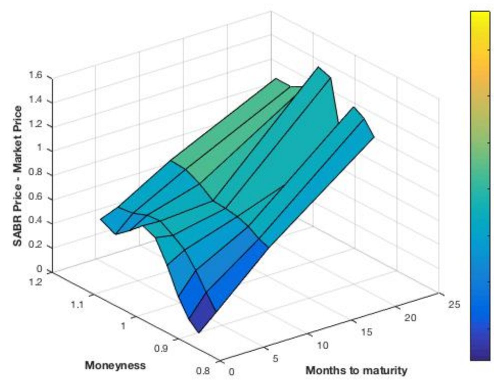
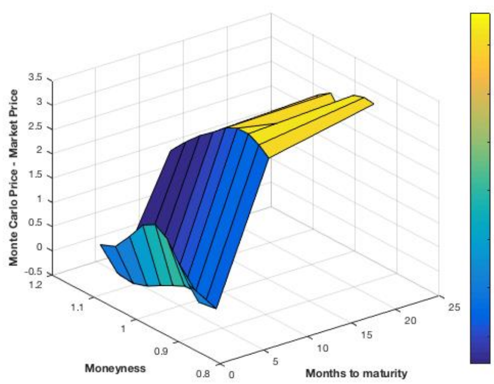


Figure 253: Exxon Mobil Monte Carlo - Market price differential ($) - Call - 11/04/16



In terms of pricing the option (Figures 251 to 253), the Heston model is the one that dominates the comparison. Indeed, the discrepancies in prices are almost null for the one-month to one-year maturities, and for the two-year maturity they approach the negative $1.5 figure. The SABR model has more and more imprecise pricing as time to maturity increases, arriving to the $1.4 figure for the call having two years to maturity.

While the Monte Carlo methodology is quite precise for short maturities, the price differential goes beyond the $2 dollar figure for longer ones.

Let us now see how the three models perform in terms of the put option:

Table 72: Exxon Mobil Mean of Implied Volatilities (%) - Put

	May/16	Jul/16	Jan/17	Jan/18
70	30.61	28.16	27.69	26.97
72.5	28.74	27.55	27.04	26.15
75	27.02	24.97	26.07	25.23
77.5	25.50	23.75	25.03	24.51
80	24.04	22.85	24.31	24.68
82.5	23.58	21.86	23.68	24.62
85	23.40	21.41	23.10	23.68

Table 73: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
May/16	0.21	0.5	-0.03	1.96
Jul/16	0.19	0.5	-0.17	1.53
Jan/17	0.21	0.5	-0.39	0.76
Jan/18	0.20	0.5	0.00	0.88

Figure 254: Exxon Mobil Market Implied Volatility (%/100) - Put - 11/04/16

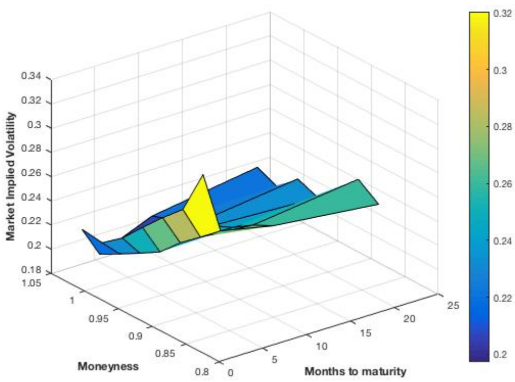


Figure 255: Exxon Mobil Heston Implied Volatility (%/100) - Put - 11/04/16

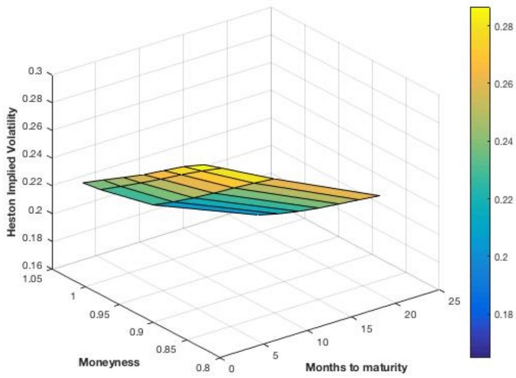


Figure 256: Exxon Mobil SABR Implied Volatility (%/100) - Put - 11/04/16

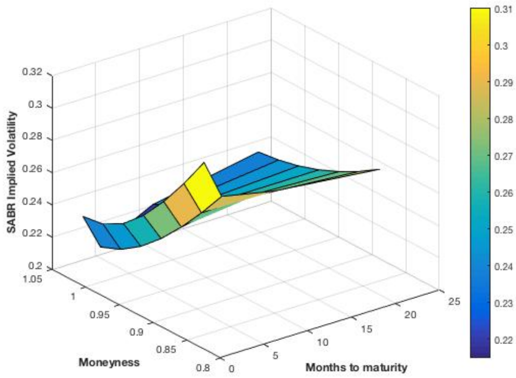
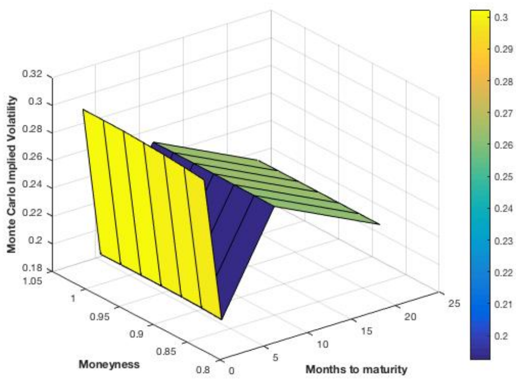


Figure 257: Exxon Mobil Monte Carlo Implied Volatility (%/100) - Put - 11/04/16



As Figures 254 to 257 show, the SABR model is the ones that most closely approaches the implied volatility shown by the market data. Indeed, it shows the spike in volatility when the option has one month to maturity and shows correctly the smile for the same maturity and skews for other times to maturity.

Although the Heston model calibrates the implied volatilities quite correctly, it fails to show the smile and the spike in volatilities when the put has one month to maturity.

The Monte Carlo process instead not only fails to show smiles and skews according to maturity, but it also overestimates it for the put having one month to maturity.

Figure 258: Exxon Mobil Heston - Market price differential ($) - Put - 11/04/16

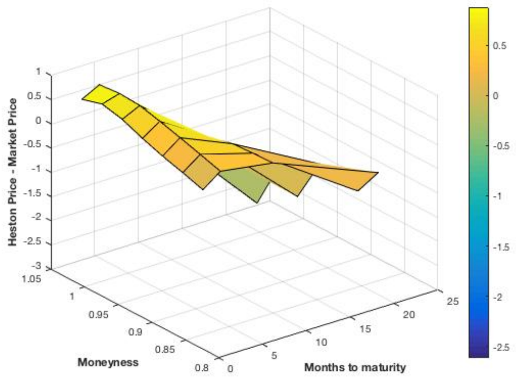


Figure 259: Exxon Mobil SABR - Market price differential ($) - Put - 11/04/16

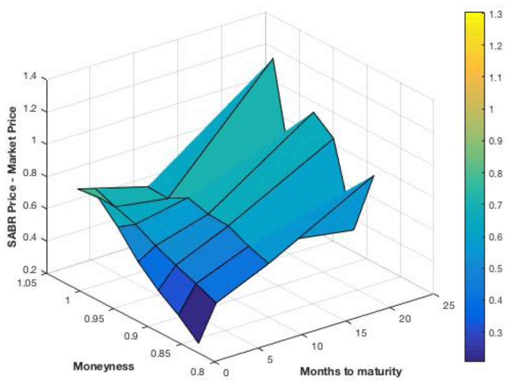
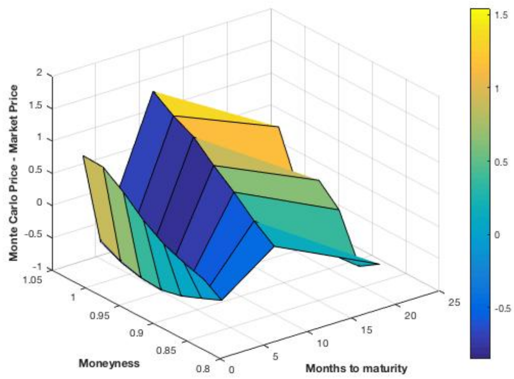


Figure 260: Exxon Mobil Monte Carlo - Market price differential ($) - Put - 11/04/16



As we can, lastly, see from Figures 258 to 260, the SABR model is the most constant one in showing the discrepancies in prices. Indeed it is quite precise for the put having a few months to maturity, and the differential goes beyond $1 for the option having two years to maturity.

The Heston model, instead, underprices the option greatly, arriving to an absolute value differential of $2.5 when the put has two years to maturity and is in-the-money.

Lastly, the Monte Carlo price differentials range from -$1 to +$1.5, depending on strike and maturity.

8.6.4 Underlying appreciation between 1% and 5%

Unfortunately, throughout March 2016 and April 2016, the stock of Exxon Mobil has not moved above 1% for a sufficient number of times (at least three) to suffice the carrying out of the analysis. Hence, no further examination will be carried out for this specific underlying price movement.

9 Attempts at capturing the dynamics of the implied volatility surface - 2

9.1 The evolution of the market: Heston, SABR and Monte Carlo implied volatility surfaces - Citigroup

The following analysis comes after having retrieved the implied volatility surface evolutions for Citigroup options maturing on May 20^{th} 2016, September 16^{th} 2016, January 20^{th} 2017 and January 19^{th} 2018. The analysis comes after observing the volatility surface under the actual market data, the Heston model, the SABR methodology and the Monte Carlo model for the following dates:

- March 28^{th} 2016
- April 4^{th} 2016
- April 11^{th} 2016
- April 18^{th} 2016
- April 25^{th} 2016

Through these data we will see how the implied volatility surface evolves under each model and how it has actually evolved (under the market data retrieved at the end of each business day reported just previously).

The analysis will both explore how the actual implied volatility surface evolves week after week and how the models' surfaces approximate the market implied volatility surface.

9.1.1 March 28^{th} 2016

The following data comes after an actual stock price decrease of 0.05%. To be clear, for this specific study we have not used the average implied volatilities, but simply the implied volatility retrieved for another previous business day where the stock has also moved between 0% and +1%. In this case, the stock price movement of been of -0.74% and it occurred on March 14^{th} 2016. The stock price has indeed moved between -1% and 0% only once between the commencement of this study and March 28^{th} 2016.

Tables 74 and 77 show the respective average implied volatilities and SABR calibrated parameters for the stock movement.

Table 74: Citigroup Implied Volatility Surface (%) - Call

	May/16	Sep/16	Jan/17	Jan/18
35	46.97	40.41	36.01	36.52
40	36.57	32.86	32.20	33.41
45	29.05	28.86	29.26	30.68
50	26.86	26.32	26.67	28.74
55	28.32	24.51	24.78	26.83

Table 75: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
May/16	0.21	0.5	-0.63	1.93
Sep/16	0.20	0.5	-0.62	1.13
Jan/17	0.20	0.5	-0.65	0.58
Jan/18	0.21	0.5	-0.61	0.47

Let us see how the market data compares to the three models' implied volatility surfaces for the call option:

Figure 261: Citigroup Market Implied Volatility Surface (%/100) - Call - 28/03/16

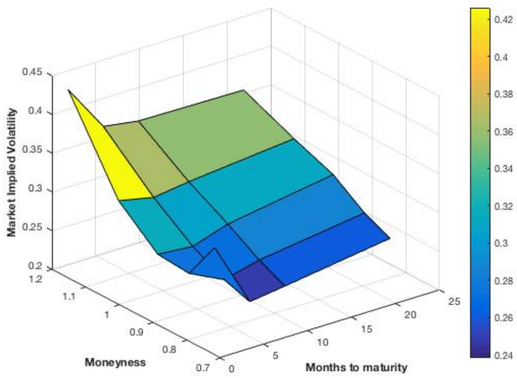


Figure 262: Citigroup Heston Implied Volatility Surface (%/100) - Call - 28/03/16

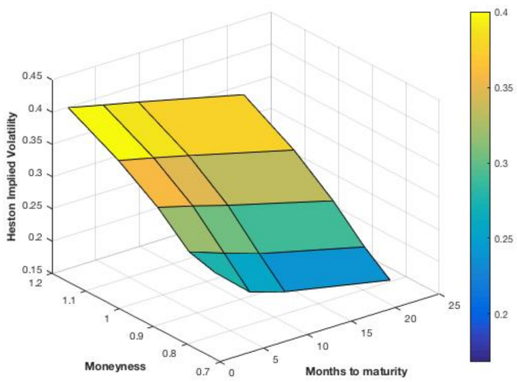


Figure 263: Citigroup SABR Implied Volatility Surface (%/100) - Call - 28/03/16

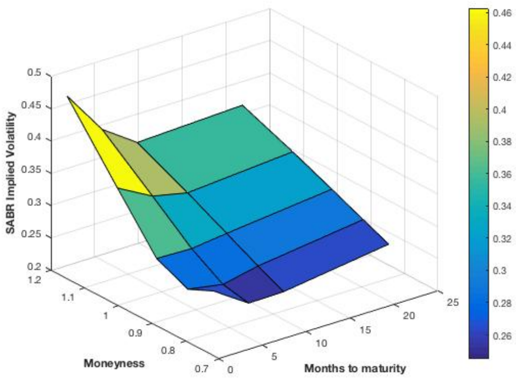
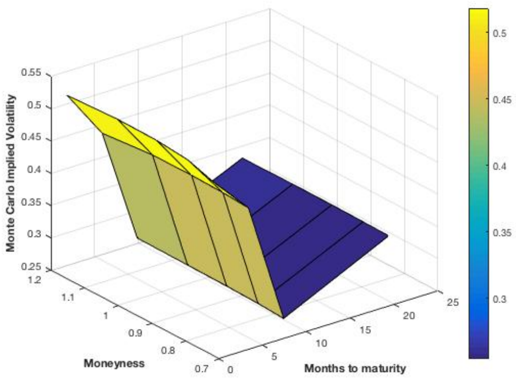


Figure 264: Citigroup Monte Carlo Implied Volatility Surface (%/100) - Call - 28/03/16



For the call option we can see that the market implied volatility surface (Figure 261) is best approximated by the SABR model (Figure 263). Indeed, it manages to capture the spike in volatility when the option is in-the-money with a few months to maturity, as well as the smile for the option having almost two months to maturity.

The Heston model (Figure 262) does a discrete job as well, except that it does not capture the spike in volatility when the call is in-the-money with a few months to maturity.

The Monte Carlo process (Figure 264) instead fails to show the smiles and skews for the respective maturities and overestimates the implied volatility when the call has a few months to maturity.

With regards to the analysis of the market implied volatility surface data, we can see that there is a smile when the option has two months to maturity and skews for the other maturities. Hence, as the option becomes more in-the-money implied volatility increases (with the exception when the call has a few months to maturity and is out-of-the-money).

Moreover, with the exception of when the call is very in-the-money, the implied volatility levels are identical for the option having two months to maturity and one year to maturity, and such levels increase as time to maturity increases.

It is interesting to see that out-of-the-money options, which are the more liquid ones, have lower implied volatility than those in-the-money.

Let us now see how the implied volatility levels compare for the put option:

Table 76: Citigroup Implied Volatility Surface - Put (%)

	May/16	Sep/16	Jan/17	Jan/18
20	67.58	59.28	53.13	47.05
25	62.11	50.20	45.90	42.04
30	49.61	42.31	39.80	36.85
35	39.84	36.40	35.11	33.46
40	33.50	31.81	31.35	30.53
45	29.79	28.13	28.14	29.29

Table 77: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
May/16	0.21	0.5	-0.55	1.33
Sep/16	0.19	0.5	-0.40	1.06
Jan/17	0.19	0.5	-0.47	0.77
Jan/18	0.19	0.5	-0.23	0.63

Figure 265: Citigroup Market Implied Volatility Surface (%/100) - Put - 28/03/16

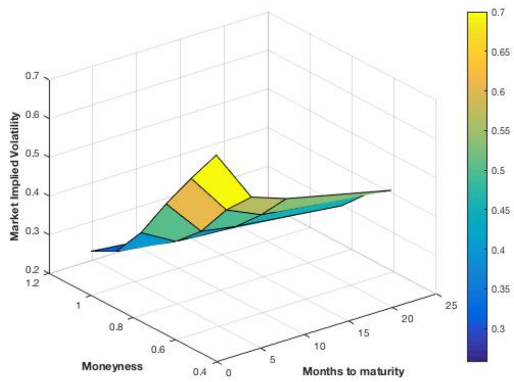


Figure 266: Citigroup Heston Implied Volatility Surface (%/100) - Put - 28/03/16

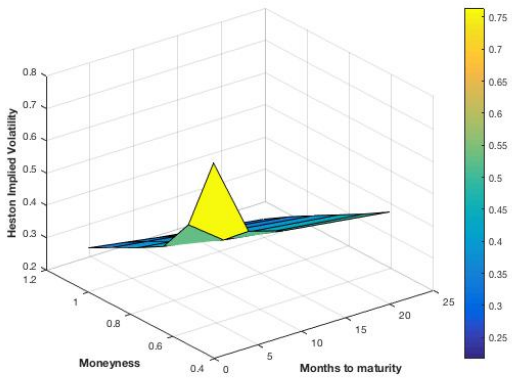


Figure 267: Citigroup SABR Implied Volatility Surface (%/100) - Put - 28/03/16

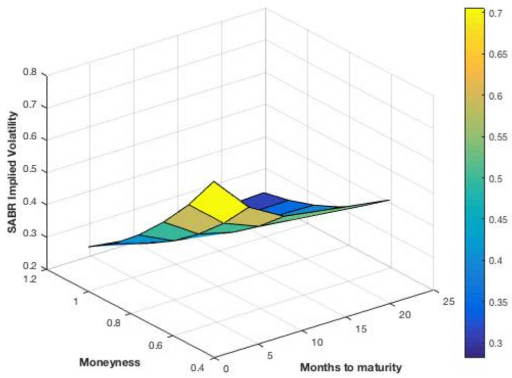
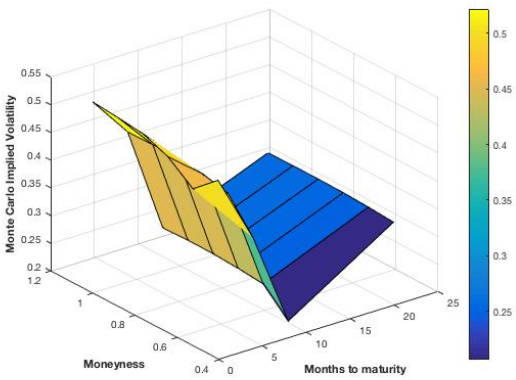


Figure 268: Citigroup Monte Carlo Implied Volatility Surface (%/100) - Put - 28/03/16



For the put as well we can see that there are skews for each maturity, but no smile this time. Both the Heston (Figure 266) and the SABR (Figure 267) models manage to capture the spike in volatility when the option is very out-of-the-money with a few months to maturity. However, the SABR model does a better job at approximating the convexity of the various curves for each maturity than the Heston model. Lastly, both models approximate the levels of implied volatility quite well when compared to the actual implied volatility data.

The Monte Carlo methodology (Figure 268), instead, overestimates implied volatility when the put option has a few months to maturity, and does the opposite when the option has two years to maturity. It is slightly more precise when the put is in-the-money in terms of showing implied volatility levels.

When analysing the market implied volatility (Figure 265), the shape of the volatility surface is similar to that of the call option, even though it must be underlined the fact that the spike in volatility when the strike price is quite low and when the option has less than two months to maturity is much higher for the put option (70%) than for the call option (slightly above 40%).

Moreover, there is no smile when the put option has less than two months to maturity, and implied volatility as a function of time to maturity decreases as the length of the option's life augments. This is also a diverse aspect when compared to the call option.

Lastly, the put option has slightly higher implied volatility levels given each strike price when compared to the call option. Indeed, the volatilities for out-of-the-money puts is far higher than that for in-the-money calls, and the same can be said when comparing in-the-money puts with out-of-the-money calls. This could be due to the fact that the market is more scared of downward moves than upward ones, and hence places a higher volatility on stocks and derivatives when the prices of such financial instruments decrease.

9.1.2 April 4^{th} 2016

For this study we will follow an underlying depreciation of 0.97% occurred on April 4^{th} 2016. For the average implied volatilities used for the Heston and SABR model, the implied volatility data has been retrieved after a stock depreciation of 0.74% occurred on March 14^{th} 2016 and another depreciation of 0.31% occurred on March 31^{st} 2016.

Tables 78 and 81 show the mean implied volatilities and SABR calibrated parameters used to find the new implied volatilities.

Table 78: Citigroup Mean of Implied Volatilities (%) - Call

	May/16	Sep/16	Jan/17	Jan/18
35	43.97	37.32	34.85	35.36
40	33.30	30.87	31.04	32.42
45	27.64	27.28	27.88	29.39
50	26.32	24.88	25.43	27.48
55	31.16	23.61	23.76	25.83

Table 79: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
May/16	0.19	0.5	-0.52	2.12
Sep/16	0.19	0.5	-0.59	1.02
Jan/17	0.19	0.5	-0.62	0.63
Jan/18	0.20	0.5	-0.59	0.52

Figure 269: Citigroup Market Implied Volatility Surface (%/100) - Call - 04/04/16

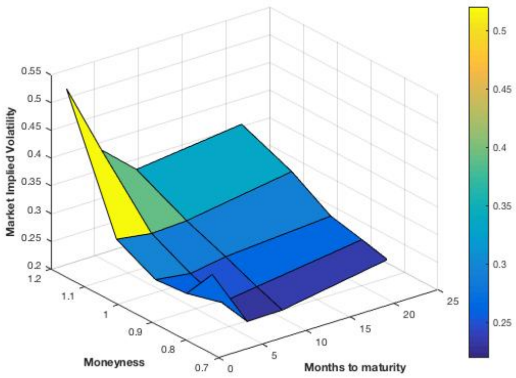


Figure 270: Citigroup Heston Implied Volatility Surface (%/100) - Call - 04/04/16

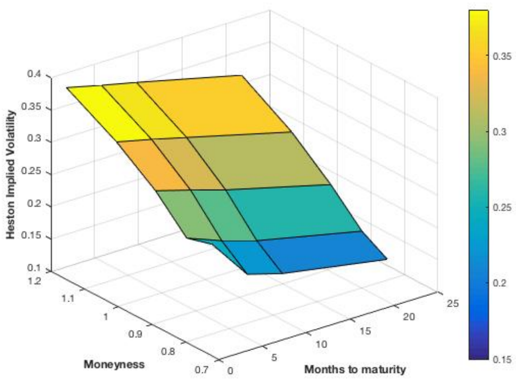


Figure 271: Citigroup SABR Implied Volatility Surface (%/100) - Call - 04/04/16

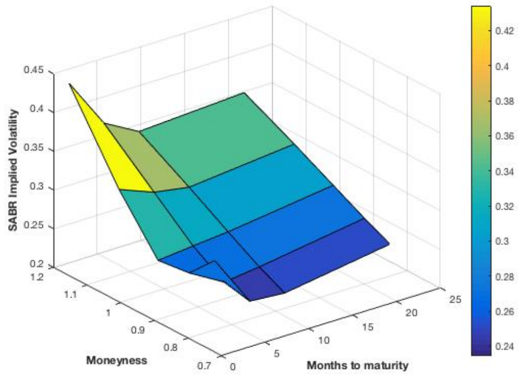
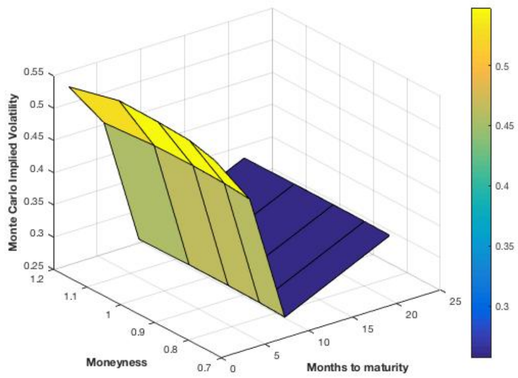


Figure 272: Citigroup Monte Carlo Implied Volatility Surface (%/100) - Call - 04/04/16



It is interesting to point out the fact that, compared to the implied volatility surfaces of March 20^{th} 2016, now that we are one week closer to maturity, both the SABR (Figure 271) and Heston (Figure 270) models approach better the smile in volatility when the call option has 1.5 months to maturity. Moreover, both models underestimate the spike in implied volatility when the option is very in-the-money with 1.5 months to maturity.

The Monte Carlo model (Figure 272) once again fails to portray the smiles and skews for the respective maturities and overestimates implied volatilities when the option has a few months to maturity.

In terms of the evolution of the market implied volatility surface form the previous week, we can see that the shape of the surface is almost unchanged (please refer to Figure X269XX), except for the fact that the spike in implied volatility for the call being in-the-money with May 20th 2016 as maturity is much more acute, passing the 50% figure (compared to that of slightly above 40% for a week earlier). For the rest of the surface, the implied volatilities have remained pretty much unchanged compared to a week earlier.

Table 80: Citigroup Mean of Implied Volatilities (%) - Put

	May/16	Sep/16	Jan/17	Jan/18
20	69.34	58.06	52.10	45.62
25	61.14	50.05	44.77	42.21
30	47.85	40.69	38.64	35.76
35	38.33	34.74	33.81	32.22
40	31.78	30.31	30.11	29.60
45	28.23	26.83	27.03	27.71

Table 81: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
May/16	0.19	0.5	-0.38	1.62
Sep/16	0.18	0.5	-0.36	1.13
Jan/17	0.18	0.5	-0.44	0.80
Jan/18	0.18	0.5	-0.41	0.57

Figure 273: Citigroup Market Implied Volatility Surface (%/100) - Put - 04/04/16

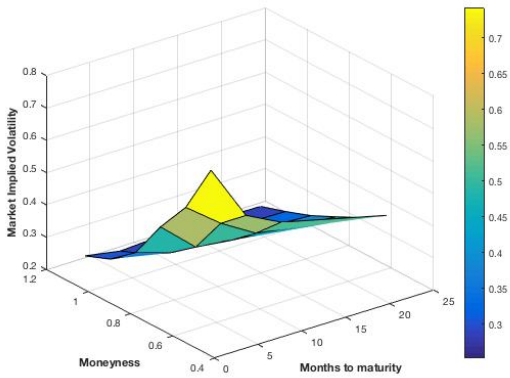


Figure 274: Citigroup Heston Implied Volatility Surface (%/100) - Put - 04/04/16

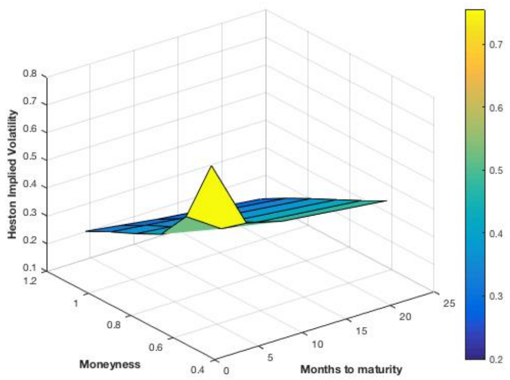


Figure 275: Citigroup SABR Implied Volatility Surface (%/100) - Put - 04/04/16

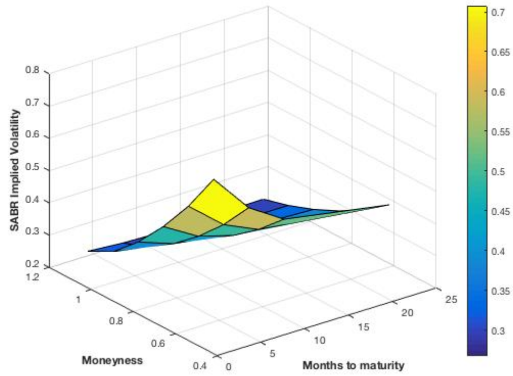
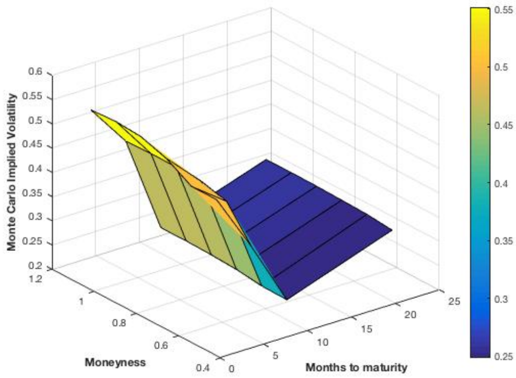


Figure 276: Citigroup Monte Carlo Implied Volatility Surface (%/100) - Put - 04/04/16



For the put option as well market implied volatilities (Figure 273) increase slightly compared to a week earlier. The peak in implied volatility is once again when the put is far out-of-the-money with 1.5 months to maturity, reaching over 70% (compared to the previous week peak of exactly 70%). Moreover, volatility decreases as time to maturity increases, as seen in the previous analysis of March 28^{th} 2016. There are volatility skews present at all maturities.

Hence, the shape of the surface has remained almost unchanged, with also the SABR (Figure 275) and Heston (Figure 274) models doing a discrete job in representing the approximations of implied volatility for all strikes and maturity (with the SABR model being slightly more accurate in showing the convexity of the skews for each time to maturity).

The Monte Carlo methodology (Figure 276) does a poor job in approximating volatility at all strikes and maturities, overestimating it once again when the put has a few months to maturity.

9.1.3 April 11^{th} 2016

Unfortunately, just one set of implied volatilities has been recorded for Heston and SABR, which comes from underlying appreciation of 1.72% occurred on April 1^{st} 2016. For the actual market data, the stock price movement of April 11^{th} 2016 has been of +1.61%.

Tables 82 and 85 show the average implied volatilities and SABR parameters for the calls and puts corresponding to this daily underlying movement.

Table 82: Citigroup Mean of Implied Volatilities - Call(%)

	May/16	Sep/16	Jan/17	Jan/18
35	42.87	34.38	34.50	34.58
40	30.57	29.24	29.87	31.18
45	25.68	25.64	26.81	28.26
50	25.20	23.39	24.11	26.33
55	30.47	22.56	22.32	24.61

Table 83: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
May/16	0.18	0.5	-0.57	2.26
Sep/16	0.18	0.5	-0.59	0.91
Jan/17	0.19	0.5	-0.66	0.70
Jan/18	0.20	0.5	-0.60	0.56

Figure 277: Citigroup Market Implied Volatility Surface (%/100) - Call - 11/04/16

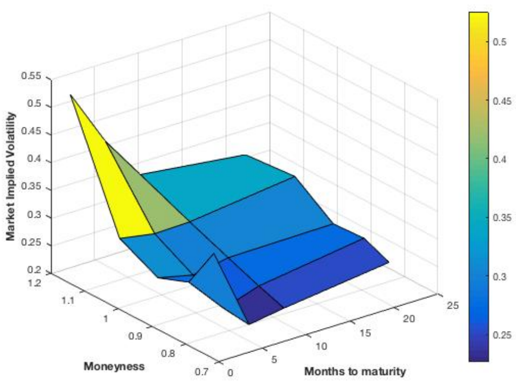


Figure 278: Citigroup Heston Implied Volatility Surface (%/100) - Call - 11/04/16

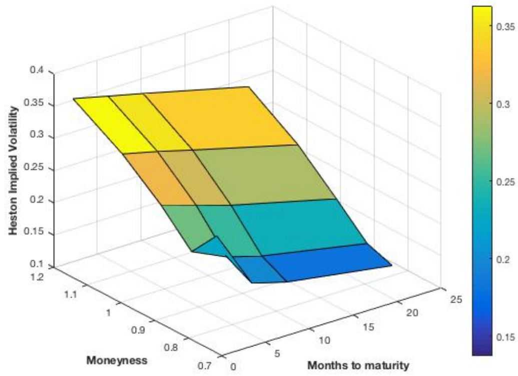


Figure 279: Citigroup SABR Implied Volatility Surface (%/100) - Call - 11/04/16

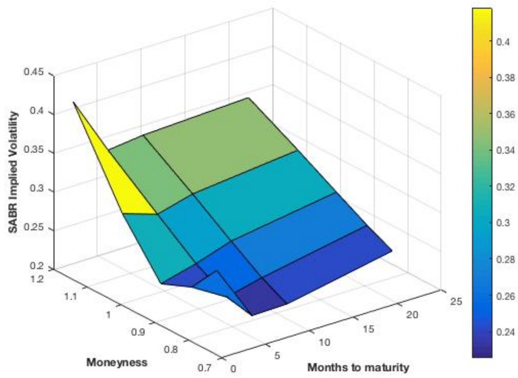
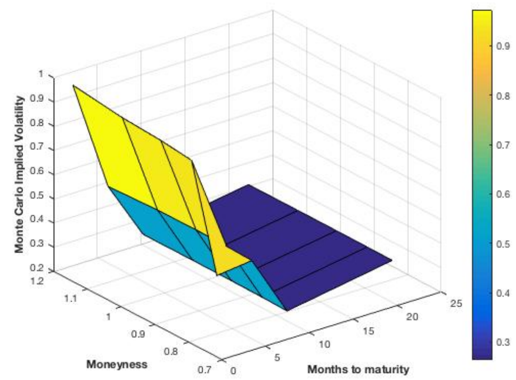


Figure 280: Citigroup Monte Carlo Implied Volatility Surface (%/100) - Call - 11/04/16



The implied volatility surface of the market (Figure 277) has been almost unchanged compared to that of the previous week, except for the fact that there is no more a skew when the call has two years to maturity. There is also a decline in volatility when the option is in-the-money with two years to maturity.

Once again, the best model to represent the market implied volatility surface is the SABR one (Figure 279), since it is the only one to portray correctly the spike in volatility when the call is in-the-money with five weeks to maturity.

The Heston model (Figure 278) is quite precise as well. However, it misses the spike in volatility that occurs when the option is far in-the-money with a bit more than one month to maturity.

The Monte Carlo (Figure 280), although it still overestimates volatility when the call has five weeks to maturity, manages to show that volatility decreases as a function of time when the call is in-the-money. However, this is not true for the call being out-of-the-money, as the market data shows.

Table 84: Citigroup Mean of Implied Volatilities (%) - Put

	May/16	Sep/16	Jan/17	Jan/18
20	72.66	57.81	51.76	45.12
25	57.81	50.49	44.19	43.14
30	48.05	39.01	37.70	34.74
35	36.62	32.89	32.64	31.45
40	29.83	28.69	28.98	28.74
45	26.22	25.46	25.97	26.41

Table 85: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
May/16	0.18	0.5	-0.35	1.86
Sep/16	0.17	0.5	-0.32	1.26
Jan/17	0.17	0.5	-0.42	0.86
Jan/18	0.18	0.5	-0.54	0.56

Figure 281: Citigroup Market Implied Volatility Surface (%/100) - Put - 11/04/16

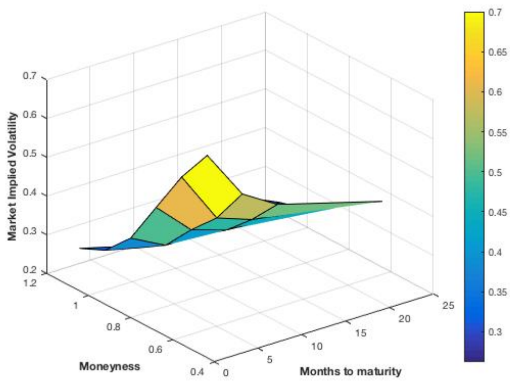


Figure 282: Citigroup Heston Implied Volatility Surface (%/100) - Put - 11/04/16

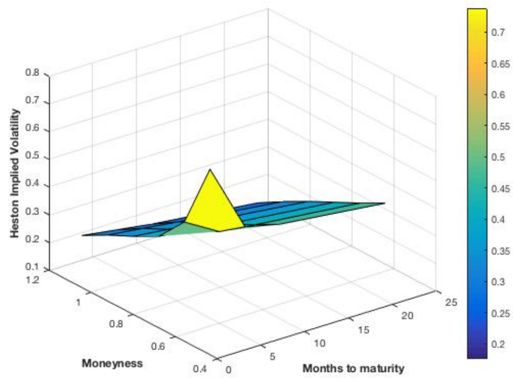


Figure 283: Citigroup SABR Implied Volatility Surface (%/100) - Put - 11/04/16

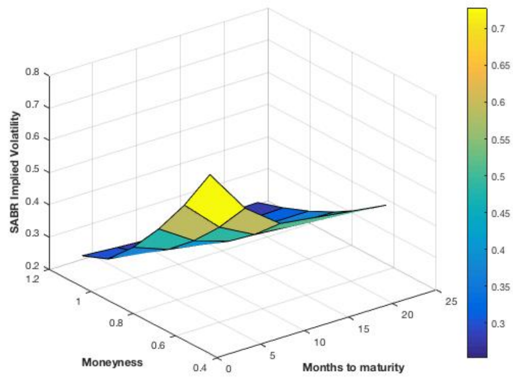
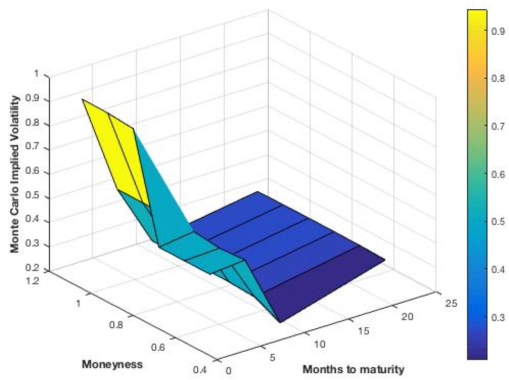


Figure 284: Citigroup Monte Carlo Implied Volatility Surface (%/100) - Put - 11/04/16



In terms of the put option (Figures 281 to 284), we can see that the skew when the option matures on May

20^{th} 2016 increases slightly when the option is at-the-money, with however the spike in volatility decreasing back to 70% when the put is out-of-the-money. Skews are still present at all maturities, and for the rest of strikes, the levels of volatility are unchanged compared to before. Here too, when the put is out-of-the-money, implied volatility decreases as a function of time to maturity.

The SABR model and Heston model manage to capture the spike in volatility when the put is out-of-the-money with a few weeks to maturity, and approximate volatility correctly for the rest of strikes and maturities. The SABR is slightly more precise in portraying the convexity of the skew when the put has a few weeks to maturity. Moreover, the levels of volatility are unchanged for both models compared to themselves one week earlier.

The Monte Carlo process, instead, arrives to place almost 100% levels of implied volatility for the put having five weeks to maturity, and understates implied volatility when the option has two years to maturity and is out-of-the-money. It fails to show the various skews according to time to maturity.

9.1.4 April 18^{th} 2016

For this specific weekly data, the averages of implied volatilities are retrieved after stock price appreciations of 0.53% on March 10^{th} 2016 and 0.38% on March 20^{th} 2016. The actual market data has been gotten on April 18^{th} 2016 after a stock appreciation of 0.42%.

Tables 86 and 89 show the average implied volatilities and SABR calibrated parameters for calls and puts for this respective daily price movement.

Table 86: Citigroup Mean of Implied Volatilities (%) - Call

	May/16	Sep/16	Jan/17	Jan/18
35	43.42	37.38	35.67	35.03
40	32.41	31.20	31.50	32.79
45	28.10	27.67	28.55	29.68
50	27.15	25.40	26.11	27.70
55	31.06	24.15	24.29	26.50

Table 87: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
May/16	0.18	0.5	-0.21	2.17
Sep/16	0.18	0.5	-0.44	1.07
Jan/17	0.19	0.5	-0.53	0.69
Jan/18	0.20	0.5	-0.51	0.48

Figure 285: Citigroup Market Implied Volatility Surface (%/100) - Call - 18/04/16

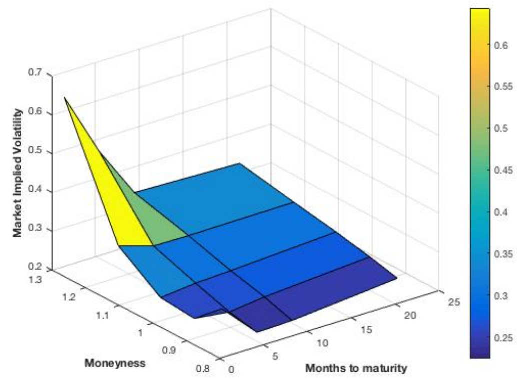


Figure 286: Citigroup Heston Implied Volatility Surface (%/100) - Call - 18/04/16

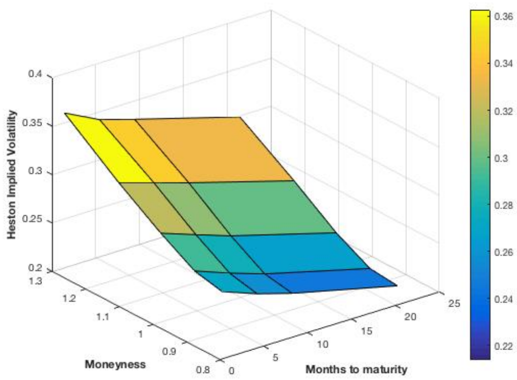


Figure 287: Citigroup SABR Implied Volatility Surface (%/100) - Call - 18/04/16

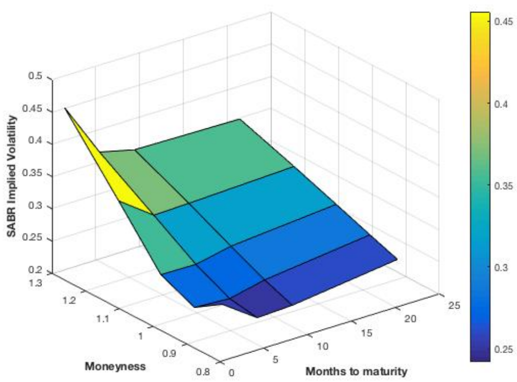
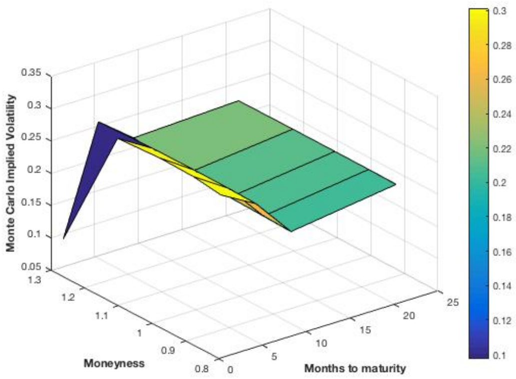


Figure 288: Citigroup Monte Carlo Implied Volatility Surface (%/100) - Call - 18/04/16



While the shape of the market implied volatility surface (Figure 285) remains unchanged, the spike in volatility when the option is in-the-money with four weeks to maturity augments drastically to over 60%. The same can be said for the call being in-the-money with a few months to maturity (implied volatility level increasing from 40% on April 11^{th} 2016 to almost 50% a week later). The rest of the volatility surface is unchanged.

The SABR model (Figure 287) not only manages to catch the spike in volatility when the option is in-the-money with four weeks to maturity more correctly than the Heston model (Figure 286), but such spike also increases compared to a week earlier, while it remains unchanged in the Heston model. However, although both models correctly estimate implied volatilities at all strikes and maturities, they still underestimate the spike in volatility when the option is in-the-money with four weeks to maturity.

Now, the Monte Carlo process (Figure 288) underestimates implied volatility, and correctly shows that implied volatility is decreasing as a function of time to maturity.

Table 88: Citigroup Mean of Implied Volatilities (%) - Put

	May/16	Sep/16	Jan/17	Jan/18
20	68.26	57.86	52.47	46.06
25	59.97	50.23	45.16	42.32
30	47.95	40.95	39.18	36.94
35	38.44	35.11	34.30	32.80
40	32.04	30.82	30.67	30.15
45	28.44	27.74	27.91	28.60

Table 89: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
May/16	0.19	0.5	-0.12	1.73
Sep/16	0.18	0.5	-0.10	1.22
Jan/17	0.18	0.5	-0.23	0.87
Jan/18	0.18	0.5	-0.32	0.57

Figure 289: Citigroup Market Implied Volatility Surface (%/100) - Put - 18/04/16

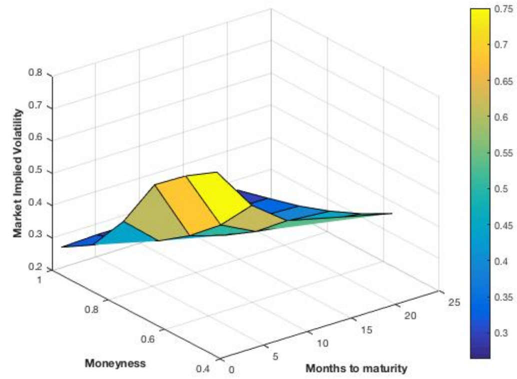


Figure 290: Citigroup Heston Implied Volatility Surface (%/100) - Put - 18/04/16

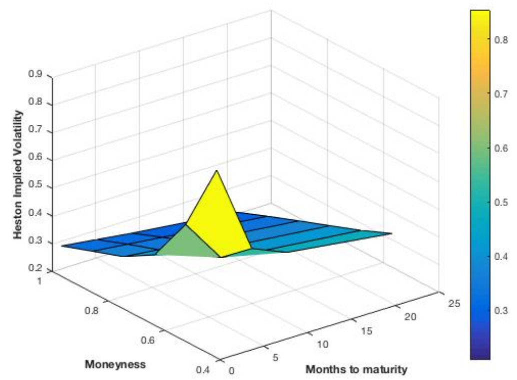


Figure 291: Citigroup SABR Implied Volatility Surface (%/100) - Put - 18/04/16

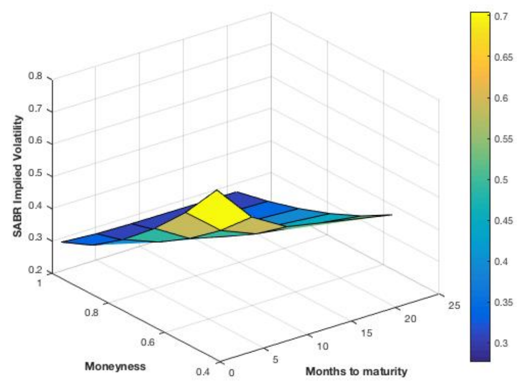
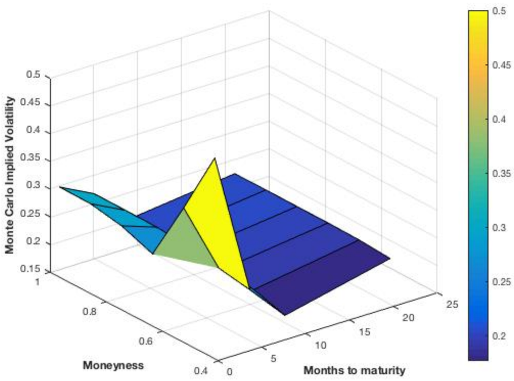


Figure 292: Citigroup Monte Carlo Implied Volatility Surface (%/100) - Put - 18/04/16



We can see that, as time passes by, the market implied volatility surface of the put (Figure 289) increases slightly in implied volatility levels. Moreover, the skews at each maturity are still present. While the SABR (Figure 291) and Heston (Figure 290) models show such skews as well, they fail to show the increase in implied volatility that occurs when the put is at-the-money with a few weeks to maturity. However, for the rest of strikes and maturities, the implied volatilities are estimated correctly.

The Monte Carlo methodology (Figure 292) is slightly more precise than in other instances, pricing in the spike in volatility when the put is out-of-the-money with a few weeks to maturity. It fails to show the skews for each time to maturity, but the misestimation in absolute value of implied volatilities are much smaller than for the previous weeks.

9.1.5 April 25^{th} 2016

For this instance, the average volatilities have been retrieved from three stock depreciations of 0.74%, 0.31% and 0.97% occurred respectively on March 14^{th} 2016, March 31^{st} 2016 and April 4^{th} 2016. For the actual market implied volatility data the stock depreciation for closing of business April 25^{th} 2016 has been of 0.60%.

Tables 90 to 93 show the respective average implied volatilities and SABR calibrated parameters for the calls and puts.

Let us now see how the models have evolved throughout this last week for the call option.

Table 90: Citigroup Mean of Implied Volatilities (%) - Call

	May/16	Sep/16	Jan/17	Jan/18
35	44.80	38.46	35.90	36.05
40	34.16	31.81	31.79	33.07
45	28.27	27.98	28.92	30.72
50	27.01	25.63	26.37	28.33
55	30.28	24.17	24.61	26.61

Table 91: SABR Calibrated parameters - Call

	Alpha	Beta	Rho	Nu
May/16	0.18	0.5	-0.13	2.26
Sep/16	0.18	0.5	-0.38	1.16
Jan/17	0.19	0.5	-0.49	0.70
Jan/18	0.21	0.5	-0.65	0.41

Figure 293: Citigroup Market Implied Volatility Surface (%/100) - Call - 25/04/16

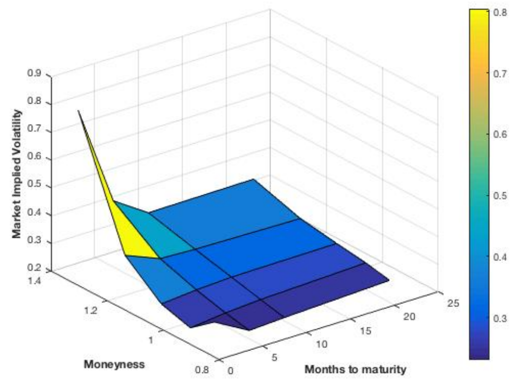


Figure 294: Citigroup Heston Implied Volatility Surface (%/100) - Call - 25/04/16

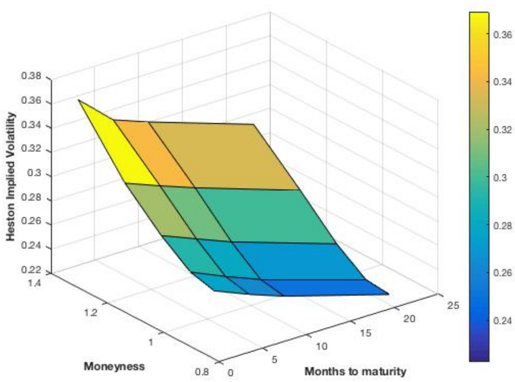


Figure 295: Citigroup SABR Implied Volatility Surface (%/100) - Call - 25/04/16

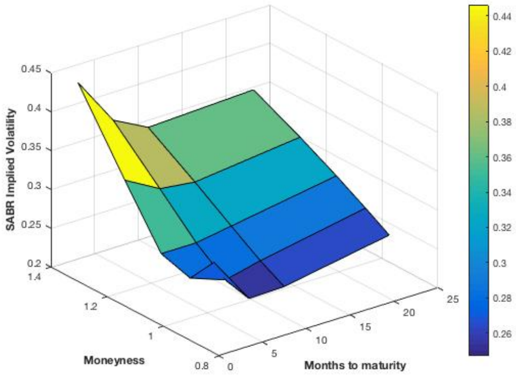
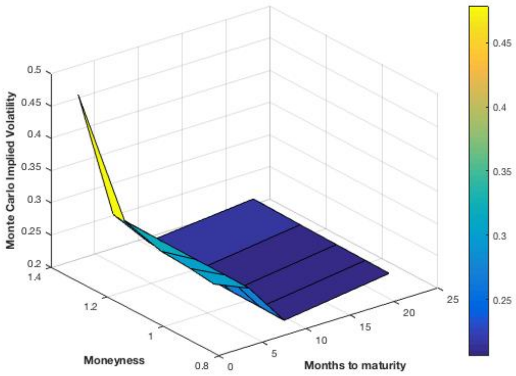


Figure 296: Citigroup Monte Carlo Implied Volatility Surface (%/100) - Call - 25/04/16



We can see that in terms of the market implied volatility (Figure 293), implied volatility stays pretty much unchanged at all strikes and maturities, with the only exception being when the option is in-the-money with one month to maturity: its implied volatility is almost 80%, vs. the almost 70% recorded a week earlier.

And it is the SABR model (Figure 295) the one that most closely matches the implied volatilities to those of the market. Indeed, it reflects the smile that is present for the call having one month to maturity, and calibrates a jump in implied volatility for the call being far in-the-money with one month to maturity.

However, this model, as well as the Heston (Figure 294) and Monte Carlo (Figure 296) models, while it manages to see that there is a jump in implied volatility, it underestimates it greatly (by about 25%).

The Heston model and SABR model manage to price in the skews for the various maturities, while the Monte Carlo fails to portray such aspect of the market implied volatility surface.

Table 92: Citigroup Mean of Implied Volatilities (%) - Put

	May/16	Sep/16	Jan/17	Jan/18
20	68.79	57.82	53.13	46.95
25	61.33	49.91	45.73	42.24
30	50.10	41.49	39.47	36.43
35	39.77	35.52	34.56	32.98
40	32.46	31.03	30.93	30.52
45	28.94	27.62	27.94	27.53

Table 93: SABR Calibrated parameters - Put

	Alpha	Beta	Rho	Nu
May/16	0.18	0.5	-0.16	1.70
Sep/16	0.18	0.5	-0.10	1.19
Jan/17	0.18	0.5	-0.16	0.92
Jan/18	0.18	0.5	-0.36	0.60

Figure 297: Citigroup Market Implied Volatility Surface (%/100) - Put - 25/04/16

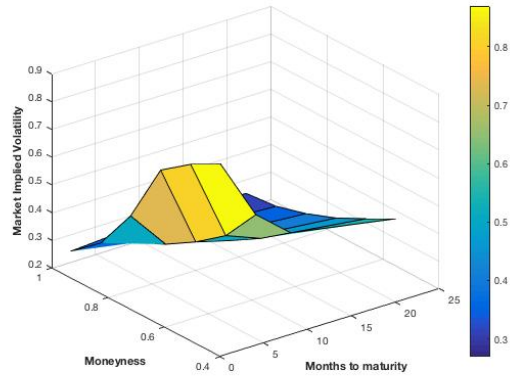


Figure 298: Citigroup Heston Implied Volatility Surface (%/100) - Put - 25/04/16

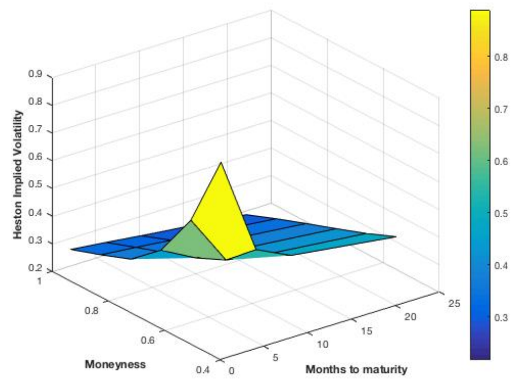


Figure 299: Citigroup SABR Implied Volatility Surface (%/100) - Put - 25/04/16

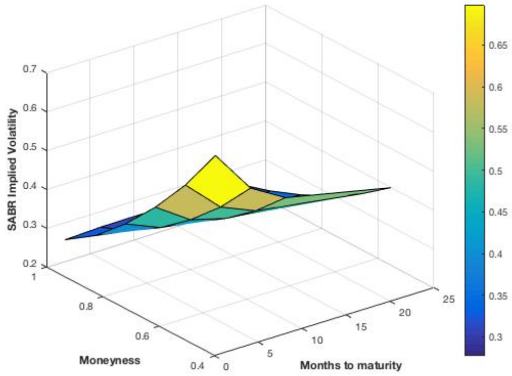
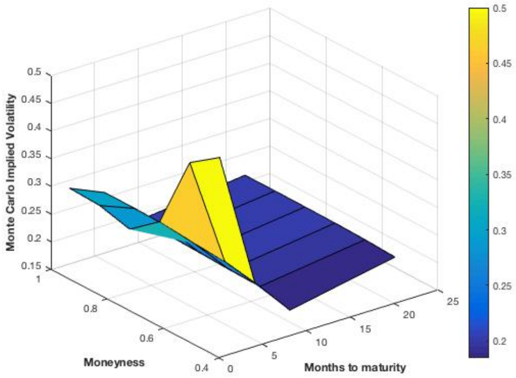


Figure 300: Citigroup Monte Carlo Implied Volatility Surface (%/100) - Put - 25/04/16



In terms of the put, it can be underlined that form observing Figure 297, market implied volatility has slightly increased for the put having one month to maturity and being out-of-the-money. Indeed, peak of volatilities arrive at 80% vs. the previous peak of 75%. The rest of the surface shows the typical skews for all maturities, with implied volatilities remaining unchanged in levels.

A surprising point is that this time it is the Heston model (Figure 298) the best one to estimate the jump in volatility for the put being far out-of-the-money with one month to maturity.

However, for strikes close to that specific strike for the same maturity, the SABR model (Figure 299) is the better model to calibrate implied volatilities.

Both models correctly show skews at all maturities.

The Monte Carlo process (Figure 300) fails to show the skews for the various maturities and, although it managed to price in a spike in implied volatility for the put being far out-of-the-money with one month to maturity, it underestimates such spike.

Therefore, it can be stated that the Heston model and the SABR model perform equally well in this instance.

10 Conclusions and further applications

After all of the previous empirical evidence, conclusions can now be drawn for both empirical studies.

10.1 Conclusions of the first empirical study

Tables 94 to 96 summarise the performances of the three models under the various circumstances that have been explored in the previous sections of this research paper.

These three Tables should be read in the following way:

- Yellow cell: the Heston model is the one that most closely captures implied volatility or that matches most closely the price to the market data

- Orange cell: the Heston model and the SABR model are the ones that most closely capture implied volatility or that match most closely the price to the market data

- Pink cell: the SABR model is the one that most closely captures implied volatility or that matches most closely the price to the market data

- Purple cell: the SABR model and the Monte Carlo model are the ones that most closely capture implied volatility or that match most closely the price to the market data

- Blue cell: the Monte Carlo model is the one that most closely captures implied volatility or that matches most closely the price to the market data

- Grey cell: the Heston model and the Monte Carlo model are the ones that most closely capture implied volatility or that match most closely the price to the market data

- Blue cell: all three models are accurate in capturing implied volatility or at matching the price to the market data

- White cell: insufficient data has been recovered in order to draw a conclusion.

Moreover, please refer to the respective Figures seen previously for an accurate reference to the conclusions that follow.

Table 94: Implied volatility capture - Empirical Study 1

	C	GS	ZION	GOOG	XOM
-5% to -1% Call					
-5% to -1% Put					
-1% to 0% Call					
-1% to 0% Put					
0% to 1% Call					
0% to 1% Put					
1% to 5% Call					
1% to 5% Put					

Table 95: Prices differentials summary - Empirical Study 1

	C	GS	ZION	GOOG	XOM
-5% to -1% Call					
-5% to -1% Put					
-1% to 0% Call					
-1% to 0% Put					
0% to 1% Call					
0% to 1% Put					
1% to 5% Call					
1% to 5% Put					

Table 96: Implied volatility surface evolution and capture - Empirical Study 2

	C
28/03/16 Call	
28/03/16 Put	
04/04/16 Call	
04/04/16 Put	
11/04/16 Call	
11/04/16 Put	
18/04/16 Call	
18/04/16 Put	
25/04/16 Call	
25/04/16 Put	

10.1.1 The implied volatility surface capture

Please refer to Table 94 and to the corresponding Figures for the conclusions that follow this section of the conclusion for the first empirical study.

The first conclusion that can be drawn is that the SABR model is the most reliable model to use to construct implied volatility surfaces that most closely match those of Yahoo! Finance. Indeed, Table 94 shows that the majority of cells are coloured in pink, which underlines the dominance of the SABR model for most instances. The Monte Carlo model, on the other hand, is never optimal to use when estimating implied volatility surfaces. It always either underestimates or overestimates volatility for all strikes and maturities, and does not portray neither smiles nor skews where appropriate. Hence, it will not be analysed further in this section of the conclusions.

However, the Heston model is a valid candidate to use depending on the situation, the company profile and the underlying movement predicted by the analyst. This is confirmed by the fact that Table 94 shows a lot of orange cells, underlining that both the Heston and SABR models show reliability in calculating implied volatility surfaces.

It is also interesting to see that the Heston model works as well as the SABR model with put options when estimating implied volatility. Indeed, most of the orange cells of Table 94 include situations in which the specific puts are analysed.

The Heston model, additionally, does not manage to show implied volatility with jumps when the option is in-the-money. It shows continuous values for volatility and fails to show big discrepancies for volatilities that are seen in the market data for calls or puts with the same time to maturity but with different strikes. The SABR model on the other hand, manages to portray such spikes in volatility when the option is both in-the-money and out-of-the-money. The Heston model manages to show such jumps in volatility when the option is out-of-the-money.

Moreover, jumps and/or series of bumps in implied volatility are present almost exclusively for very short-dated options. For longer-dated options, the estimations of volatilities are more precise for both the Heston and the SABR model.

Both the Heston and SABR model fail to show bumps for implied volatility for the same time to maturity when it is present in market data, and show instead more linearly convex curves. Such bumps in volatility for a given time to maturity are very rare, however, as the data has shown.

The Heston and SABR models correctly calibrate the convexity of skews and smiles for any give time to maturity, independently of the company or of the option being analysed.

For short-dated options, it is almost always optimal to use the SABR model for put options, while for calls 20% of the times just the SABR is the preferred method, while for the rest of the 80% of the times the Heston model is. For long-dated options, it is always the case where both the Heston model and the SABR model perform equally well.

For small changes in the underlying (for changes between -1% and +1%), it is highly recommended to adopt the SABR model to match implied volatility to that of the market. Table 94 indeed shows that 10 out of 16 cells are filled in pink and 15 out of 16 are filled in pink or orange.

For large changes in the underlying price (with an absolute value between 1% and 5%) if only one model can be used, then the SABR model is the one to adopt. However, the Heston model can be reliable too, since out of 16 cells in Table 94, 9 are orange (4 for calls and 5 for puts). The Heston model is as precise as the SABR one when analysing investment banks (such as Goldman Sachs) that have big underlying price depreciations and when analysing commercial or global banks (such as respectively Zions Bancorporation and Citigroup) with big appreciations in the underlying.

For non-bank companies it is optimal to use the SABR model to estimate implied volatility, as seen in Table 94: for Google it is almost always optimal to make reference to the SABR model (5 cells out of 6 being are pink coloured), whereas for Exxon Mobil it is a "dominant" model to use 67% of the times.

With regards to banks, if the bank has commercial banking activities in its portfolio and the underlying appreciation is above +1%, then the analyst should be indifferent between using the Heston model and the SABR model. This is according to the results for Citigroup and Zions Bancorporation shown in Table 94. If the bank has exclusively investment banking activities, then for underlying movements above the -1% figure it is better to use exclusively the SABR model to calibrate implied volatility surfaces. This is the result of the analysis of the option implied volatility surface behaviours of Goldman Sachs. Overall, however, the analyst should prefer the SABR model to the Heston one.

For very large market capitalisation ($500 billion or over) and/or Information Technology companies the SABR model is the optimal one to use in terms of estimating implied volatility. This comes form the results shown by Google.

For middle cap companies (such as Zions Bancorporation) the SABR model is still the optimal model to use to estimate implied volatility, although for large movements in the underlying price the Heston model does a discrete job as well. For positive movements in the underlying, the Heston model is preferred to the SABR model (please refer to the group of one yellow cell and three orange cells in Table 94).

For Oil & Gas firms, the SABR model is the preferred methodology to adopt. For calls, however, the Heston model can be of equal accuracy too.

With regards to the models themselves, the Heston model performs better for out-of-the-money options for commercial banks and instead for in-the-money options for investment banks. For companies belonging in other sectors, the Heston model can work well with either out-of-the-money options, in-the-money ones or both. SABR model-wise, for non-banks it works well for both in-the-money and out-of-the-money options, while for mid-caps it is best used for out-of-the-money options. For large banks, it is optimal to use it for both in-the-money and out-of-the-money options.

It is interesting to also analyse the performance of the models after specific catalysts concerning the companies are released and see if there is any specific pattern between the two.

For Citigroup, on April 7^{th} the bank "revises executive pay plans, wth proxy advisors unsatisfied"[228]. This might have led partially to the stock depreciation of 3.80% recorded in that day, and to the slight dominance as a result of the SABR model compared to the other two (one pink cell in Table 94 and one orange cell for movements in underlying between -5% and -1%). On April 14^{th} 2016, one day before the Earnings report of the bank, a comparable company to itself, Bank of America, reported an 18% drop in profits because of its trading division[229]. This might have led

[228] http://finance.yahoo.com/news/citi-revises-executive-pay-plan-205808263.html

[229] http://finance.yahoo.com/news/bank-america-profit-drops-18-112937514.html

investors to shift their money to other stronger banks such as Citigroup might have been, which has led to a stock appreciation for that day of 1.65%. However, even with this catalyst, the Heston and SABR model have performed equally to each other.

For Goldman Sachs, on April 7th 2016 the stock depreciation of 3.08%, with consequent equal dominance of the Heston and SABR model might have been determined by the fact that stock have been decreasing in value because of both Oil and the Dollar depreciating in value[230]. On April 8th 2016 it is found that "Goldman cut pay for executives in 2015"[231], and the SBAR model has beat the Heston and Monte Carlo model in terms of estimating implied volatility. On April 14th 2016 Goldman Sachs declares that it will move forward with cost cuts[232] and the poor performance of the Bank of America Earnings is released, as previously stated, which might have conditioned the fact that the SABR model has been the best performing one with respect to the Heston and Monte Carlo ones. On April 19th 2016, the investment bank posts poor revenue results[233] and consequently the SABR model performs well compared to the other two construction methodologies.

With respect to Zions Bancorporation, as stated previously, the poor performance of Bank of America in its Earnings report might have affected the small commercial bank, since its stock appreciated by 0.99%. Moreover, in this specific instance the Heston model has performed better than the SABR one for the call and equally for the put option.

Google-wise, on April 1st 2016 two major catalysts might have conditioned the performance of the models: the fact that Citi cut Google price targets[234] and the suspension of NYSE trading of the stock[235]. This might have caused the fact that the SABR model beat the other two models in terms of estimating the implied volatilities of calls and puts. On April 4th 2016, Google wins a specific court ruling over push notifications, saving itself an almost sure fine of $85 million[236]. This might have conditioned the models, making the SABR one the dominant one. On April 26th, there has been news of an examination by the FTC over Google's possible Android dominance abuse[237], which has led to a stock depreciation of 2.08% and to the Heston model performing slightly better than in other instances.

For Exxon Mobil, on April 1st 2016 the big catalyst consists of Oil prices declining sharply, and this might have affected the fact that the the SABR model is the one that performs the best for estimating implied volatilities of put options. On April 7th, there are again very weak Oil prices, and this however has led to the fact that the SABR model has once again dominated in estimating implied volatilities for puts.

Passing on to a closer analysis of the various Figures shown before in this research paper, we can see that in terms of implied volatility calibration for out-of-the-money options, the SABR model is almost always the best model to use for calibrating puts. For half of the instances for calls, both the Heston and SABR models are optimal, while for the other half of instances the SABR model is the best one to use. With regards to in-the-money options, the SABR model is almost always the preferred model to adopt for calls, 70% of the times it is also optimal for put implied volatility calibrations (the other 30% being both Heston and SABR).

10.1.2 The pricing differentials

For the analysis for the second and last section of the first empirical study, please refer to the specific Figures showing the various price differentials and to Table 95.

The first thing that comes to the eye is that there is a prevalence of pink cells in Table 95. Hence, we can state that as a general model to use to estimate option prices, the SABR model is the most accurate one.

However, we can spot a lot of yellow cells, especially for the column of Zions Bancorporation: this means that the Heston model is also an optimal model to use, depending on the situation, the company profile and sector it is involved in, and the underlying movement predicted by the analyst. Moreover, it seems that the Heston model is the most precise methodology to use for put options of banks when the underlying movement for the trading day is between -1% and 0%.

When analysing jumps in implied volatility, although they are not many and are not present often for one given implied volatility surface, there is a slight dominance of the SABR model with respect to pricing calls and puts, although for specific instances it happens that either the Heston, Monte Carlo or all three models are the respectively the most precise and equally precise. With respect to bumps in implied volatility for a given maturity, which are even rarer than jumps, there is also here a slight dominance for the SABR model, although the Heston and

[230] http://finance.yahoo.com/news/stocks-retreat-as-oil–dollar-push-lowe-191735867.html

[231] http://www.marketwatch.com/story/goldman-cut-pay-for-executives-in-2015-2016-04-08?siteid=yhoof2

[232] http://www.bloomberg.com/news/videos/2016-04-15/goldman-said-to-seek-deep-cost-cuts?cmpid=yhoo.headline

[233] http://finance.yahoo.com/news/goldman-sachs-profit-slumps-fourth-114645384.html

[234] http://www.investors.com/news/technology/citi-lowers-amazon-nflx-google-price-targets-on-stock-compensation/?ven=YahooCP&src=AURLLED&ven=yahoo

[235] http://www.marketwatch.com/story/nyse-suspends-trading-of-amazon-alphabet-6-other-securities-2016-04-01?siteid=yhoof2

[236] http://www.siliconbeat.com/2016/04/04/google-beats-back-patent-troll-saves-85-million/

[237] http://blogs.barrons.com/techtraderdaily/2016/04/26/google-slips-ftc-probe-possible-android-dominance-abuse-says-wsj/?mod=yahoobarrons&ru=yahoo

Monte Carlo model calibrate better prices for specific instances too.

With regards to short-dated options, the the Heston model does a better job at estimating option prices, and the SABR model and Monte Carlo one dominate the comparison rarely. For long-dated options, the best model to calibrate prices is the SABR model, although the Monte Carlo methodology can be quite accurate too.

Regarding out-of-the-money options, the SABR model is the most precise price calibrator, although in certain circumstances the Heston model is equally precise. For in-the-money options, the SABR model is once again the best model to use, even if the Heston and Monte Carlo models are sometimes more accurate.

By looking at Table 95, we can also see that for small changes in the underlying movement (movements within the absolute value of 1%) the SABR model is still the dominant one (13 cells pink out of 20). When analysing bigger changes in the underlying (specifically with an absolute value between 1% and 5%) there are only 8 cells pink out of 16. However, there are also an orange and a purple cell, which respectively lets us state that the SABR model is optimal for both cells. So, in this instance the SABR model is the optimal model to use 63% of the times. The Monte Carlo process is actually optimal a couple of times, and the Heston model beat the other two models 19% of the times. So, all in all the SABR model is still the recommended model to use.

With respect to the models themselves, the Heston model is more precise for in-the-money calls and out-of-the-money puts, while the SABR and Monte Carlo methodologies are better at estimating prices exclusively for out-of-the-money options.

For non-bank companies, evidence is that the SABR model should be used. Indeed, for only two cells out of 12 is the SABR model not optimal to use. It is especially advised to adopt when the underlying movement is less than 1% in absolute value. When analysing banks, there is more of a mix in the optimal models to use, with the Heston methodology being optimal in various instances. Overall, however, the SABR model is still the one that constitutes the best one in pricing options, with 12 cells out of 24 being pink and one cell being orange.

Regarding investment banks, such as Goldman Sachs, the SABR model is the optimal model to use when the underlying price movement is positive. However, when such movement is negative in percentage, both the SABR and Monte Carlo models are the best models to use. The final answer will depend on whether the underlying movement is above or below -1% and whether the analyst wants to predict the prices of calls or puts. Commercial banks-wise, the SABR model is especially great to use for positive changes in the underlying price, while for negative changes in the stock price the Heston model is the most accurate.

With regards to Information technology companies, such as Google, the SABR model is the most accurate model to use when pricing options, except when pricing calls with underlying movements of less than -1%. For Oil & Gas firms, the SABR model is the most precise one to use, as evidence from the Exxon Mobil analysis shows.

For large cap companies, the SABR model is the one that most closely matches option prices to those of the market. However, for mid-cap firms, such as Zions Bancorporation, Table 95 shows that the Heston model is the most appropriate one to adopt, especially when the movement in the underlying price is negative.

Let us now explore whether there is a pattern between specific catalysts and performances of the models. Let us retake the same catalysts that have been listed in the previous section of this conclusion part.

On April 7^{th} Citigroup does not satisfy proxy advisors for the revision of executive pay plans, and the result is that the SABR model is the optimal one to use for calls, while there is indifference between the Heston model and the SABR one for pricing puts. For the April 14^{th} catalyst, which consists in Bank of America's poor Earnings report, the SABR model is the one that most accurately prices both calls and puts.

For Goldman Sachs, the April 7^{th} 2016 catalysts include drops in the values of both the Dollar and Oil, which has led to the SABR model dominate for pricing calls and the Monte Carlo one for puts. On April 8^{th} 2016, the cut of pay for executives might have led to the Heston model being the best model to use overall, even though for calls, the three models analysed in this research paper have been equally accurate. On April 14^{th} 2016, after the Bank of America Earnings report has been released, the SABR model has performed far better than the other two. Lastly, on April 19^{th}, after disappointing revenue results from the American investment bank, once again we are in a situation where the SABR and Monte Carlo models price most accurately respectively puts and calls.

Regarding Zions Bancorporation, the Bank of America Earnings report might have led to the fact that the SABR model has performed best for puts and the Monte Carlo model for calls.

With respect to Google, the two major catalysts of April 1^{st} 2016 (the Citi rating cut and the NYSE trading suspension) might have led to the fact that the SABR model has dominated the pricing of both calls and puts. On April 4^{th} 2016, the court ruling winning of Google might have influenced the SABR model to be the best one to adopt to price options. Lastly, on April 26^{th} 2016, the FTC examination over Google's possible Android dominance abuse might be the cause for the Monte Carlo being the most appropriate model to use to price calls and the best one, along wit the SABR model, to price puts.

Lastly, for Exxon Mobil, the April 1^{st} 2016 drop in Oil prices might be the reason why the SABR model performed better than the other two in pricing calls and puts. The other big Oil prices drop on April 7^{th} 2016 might also explain why the SABR model is once again the best pricer among the three models analysed in this research paper for calls and puts.

Additionally, the three implied volatility construction methodologies are better used with short-dated options in general when calibrating option prices.

Also, there are three important aspects to consider when using these models as option pricers: bid-ask (and hence arbitrage opportunities) prices, transaction costs and liquidity that the specific model would bring to the market. The way the conclusions are inferred from these specific analyses are in the following format: form the point of view of a price taker, the most appropriate model for pricing bid prices is that which shows the highest price differential, and vice-versa for the ask prices estimation. The opposite is true for market makers. Indeed, the idea behind the following conclusions is that if the specific investor is a price taker, then he/she buys at the ask and sells at the bid. Therefore, the investor wants a model that shows a 0-to-positive price differential for bids (meaning that he/she estimates that sell prices could be higher than what is shown in the market) and that he/she wants a model showing a 0-to-negative price differential for asks (because he/she is estimating the prices at which the options would be bought, and negative differentials mean that the investor wants to pay less than what is shown by the market). While this is not the best approach to promote market liquidity (since bid-ask spreads would skyrocket), it would help the investor avoid being exploited through arbitrage opportunities. Instead, if the analyst is examining the model which would promote market liquidity and transactions the most, then he/she should chose the model which has the lowest price differentials with those shown by the market.

Having stated these specific methodologies used to evaluate the various models, we can now state that for bids form the point of view of the client (asks from the market maker's perspective) the SABR model is the most appropriate model to use to estimate prices, while for ask prices from the point of view of the client (bid prices form the market maker's perspective) the winner prize goes equally to the Heston and Monte Carlo model. Instead, the most precise model, which would hence promote liquidity in the market, is the SABR model.

As it could have been expected by the reader, the results of this second empirical study are absolutely dependent on the following factors: the type of company, the sector it is involved in, the option being analysed, whether the option is in-the-money or out-of-the-money, the time to maturity of the option, the stock price change in value and also potential firm catalysts.

Lastly, one of the most important conclusions that can be drawn by Tables 94 and 95 is that a cell with strike X, time to maturity Y and colour Z in Table 94 does not necessarily match in colour with the respective cell with strike X and time to maturity Y of Table 95. This happens only for 44% of the times. This means that only 44% of the times the model that most correctly estimates the implied volatility surface also manages to be the most accurate option pricer.

10.2 Conclusions of the second empirical study

Please refer to the Figures of section 10 of this research paper to appreciate the conclusions that follow in this specific Section.

The first conclusion we can draw from looking at Table 96 is that the most accurate model to capture the implied volatility surface of the market throughout the evolution of time is the SABR model. Indeed, eight cells out of ten are pink and two out of ten orange. So, it is always convenient to use the SABR model.

However, we can see that as time evolves eventually also the Heston model becomes a great model to use when estimating the implied volatility surface of put options, as the two orange cells of Table 96 show. Hence, we can state that as the market implied volatility surface evolves, the Heston model approaches the SABR model in terms of precision of capturing such surface.

Before noting other conclusions, let us state that it is never optimal to use the Monte Carlo method, as the respective Figures show. From this moment onwards, the conclusions will put at comparison only the Heston and SABR model.

Bumps-wise, they are only present for Figure 277 for long-dated calls, and both models perform equally in terms of portraying those volatilities. They indeed both miss out in capturing those bumps.

For short-dated options, it is always more appropriate to use the SABR model. However, for long-dated ones both models perform equally in terms of capturing the volatility surface.

Moreover, both models are more precise to use when calibrating long-dated option volatility surfaces.

The Heston model is more accurate for out-of-the-money options. In the first weeks of this study the SABR model was precise for both in-the-money and out-of-the-money options, while as the volatility surface has evolved throughout time the construction methodology became more precise for out-of-the-money options.

For small changes in the underlying (with absolute value of less than 1%) that occur far from the maturity of the options, the SABR model is the most precise model to use. As we approach the maturity of the options, however, the Heston model becomes as precise as the SABR model. On the other hand, for bigger changes in the underlying price (changes in absolute value between 1% and 5%) it is only the SABR model to appropriately calibrate the implied volatility surface of the market.

With respect to smiles sand skews, the SABR model is the one that correctly calibrates them. Indeed, for calls there is a smile for short-dated options and a skew for longer-dated options that are correctly captured by the model. The skews part of the implied volatility surface is also well approximated by the Heston model. With regards to puts, instead, there are only skews for each respective maturity, and both the Heston and SABR model correctly estimate that there are skews for each given maturity.

With respect to which model is the best one to approach implied volatilities for out-of-the-money options, we can spot that for calls it is the SABR model, while for puts the two models perform equally. There is indeed a jump for out-of-the-money puts, which is properly calibrated in both models. Instead, for in-the-money options it is exclusively the SABR model to approach properly the implied volatility surface shown by the market. Indeed, for in-the-money calls there is a jump in volatility as the call is very in-the-money, and such jump is captured only by the SABR model.

Additionally, since the analysis has been carried out to the Citigroup options, we can state that the SABR model is the recommended model to use if the stock that of a global bank or that of a large-cap firm. However, as we approach the maturity of the short-dated options, the Heston model becomes optimal too.

In terms of the evolution of the implied volatility surfaces, we can spot that for calls the volatility surface remains unchanged except for the vary in-the-money calls with one month to maturity, where the implied volatility levels constantly increases from 40% to almost 80%. Instead, for the Heston and SABR models the implied volatility surface does not change with time passing by, making them both miss out on the spike in volatility that occurs for the calls being in-the-money with one month to maturity. With respect to puts, the market implied volatility surface remains unchanged for all strikes and maturities, except for out-of-the-money and at-the-money puts, where implied volatility levels evolve respectively from 70% to 85% and from 40% to 70%. The Heston model also has an unchanged implied volatility surface for puts, with the exception of out-of-the-money ones where it correctly estimates an increase of implied volatility throughout time from 70% to 80%. The SABR model also has an unvaried implied volatility surface for all strikes and maturities throughout time, except the spike in volatility for out-of-the-money puts which increases in value from 40% to 70%. Hence, it is interesting to point out that for puts the Heston model is the most precise in capturing spikes in volatility. Furthermore, it can be added that both models are more precise when the options are far from maturity than otherwise. Indeed, they do not capture the increase in volatility for in-the-money calls with one month to maturity and slightly underestimate implied volatility for out-of-the-money puts as weeks pass by.

Let us now analyse how potential catalysts might have conditioned the performance of the Heston and SABR models in estimating implied volatility surfaces.

On March 28^{th} 2016, the major headline was that "Stocks are on edge as Fed commentary, job report looms"[238]. This might have caused the SABR model to be more precise with respect to the Heston model. On Monday April 18^{th} 2016 there is a downgrade to Citigroup stock[239], which might have led to the SABR model being the best model to use for implied volatility surface estimation.

Lastly, it can be stated that results do vary depending on whether the option being analysed is a call or a put, as this Section has shown just now.

10.3 Advantages and limitations/zones of improvement for this study

10.3.1 Advantages of the study

There are various advantages, limitations and zones for improvement for the research that has been carried out in the empirical section of this paper.

Let us start by stating the advantages of such methodology:

1 - There is the relaxation of many assumptions of the Black-Scholes model, such as constant implied volatilities and constant rates.

2 - The models are based on reasonable inputs, such as rates of 0.25% and dividend rates according to the specific companies dividend policies. Moreover, the initial implied volatility input is taken from the average of previously observed implied volatilities, which means that actual, reasonable data is used to estimate implied volatility surfaces and option prices.

3 - Moreover, with regards to the Heston model, it allows for a "closed form solution for the European options"[240], for "non log-normal probability distributions and it fits the implied volatility surface of the options

[238] http://finance.yahoo.com/news/stocks-on-edge-as-fed-commentary–jobs-report-looms-191358208.html

[239] http://www.marketwatch.com/story/citigroups-stock-drops-after-analyst-downgrade-2016-04-18?siteid=yhoof2

[240] http://pure.au.dk/portal-asb-student/files/8505/222384.pdf

prices in the market"[241]. Additionally, the volatility of the model experiences mean reversion and "permits the correlation between the asset and the volatility to be changed"[242].

4 - The models can produce "realistic dynamics, such as forward volatility"[243].

5 - The models are relatively quick in measuring implied volatility through the MATLAB 2016 code.

6 - Moreover, for very short-dated options the models give relatively precise prices. Hence, precision for specific options and specific situations is another advantage of the three models used in this study. The Heston and SABR models are also relatively precise for calibration of implied volatility surfaces.

7 - This methodology allows to see right away which method is the best one under which circumstance through the development of the 3D figures.

8 - The study also allows the analyst to understand where each model can be improved or used. For example, the Monte Carlo method is never optimal to calibrate implied volatility surfaces, so it should not be used further in future studies.

9 - Such MATLAB 2016 and Excel codes provide the basis for further, more sophisticated studies. For example, we could use a combination of the SABR and Heston model with the addition of jumps in stock price and volatility would allow the analyst to develop a SVJJ model.

10 - The study can be further applied to options of any asset class as long as the underlying has a sufficient amount of implied volatility points for a given surface.

11 - The models can be used for exotic options. With regards to American options, only the Heston and SABR models can be used.

10.3.2 Limitations of the study and zones for improvement

Let us now see the limitations and zones for improvement for the study that has been carried out in the empirical section of this research paper:

1 - The most important limitation of this study is that past results might not necessarily represent what will happen in the future for specific stocks and underlying movement percentages.

2 - Correct implied volatility surface estimation does not necessarily lead to precise option pricing, and viceversa. Therefore, discrepancies between the models prices and the actual market data might be due to factors other than implied volatility, such as interest rates, dividends, frictions, etc.

3 - Even if a model is more precise than the others for option pricing, for instance, it does not mean that the model is necessarily precise itself.

4 - The implied volatilities and prices of options from these models are extremely sensitive to the parameters decided beforehand by the analyst.

5 - Some of the Heston and SABR parameters have been chosen before hand and it has been chosen to keep them fixed. An example is the SABR model's Beta which has been set to equal 0.5. However, was this the best choice? Is it correct to have a fixed Beta in the SABR model or fixed interest rates in the Heston model? Improvements would include models that account for parameters that are exclusively stochastic.

6 - The codes can be improved for each model, ranging from making them more "automatic" (with hence less data to copy out manually from the Yahoo! Finance website, An example could be to create an Excel macro to extract data directly from Yahoo! Finance) to more sophisticated.

7 - What is the definition of "jump"? Does it consist in an underlying price movement of 5% move in maximum ten seconds? Or a move of 10% in maximum one minute? The definition of "jump", therefore, depends on the model that the analyst is using and a great deal of bias can result from the ambiguity of such definition.

8 - All these conclusions are based on the fact that both the Heston and SABR model initial volatilities are simply averages of previously recorded implied volatilities. In fact, "volatility is unobservable, and the parameter values are therefore not easily estimated"[244]. An improvement to this study could have been if GARCH model were used instead of simple arithmetic average of implied volatilities.

9 - The recordings in the underlying price movement used to find the average implied volatilities do not necessarily match the underlying price movement recorded when the actual market data is retrieved. For example, for Citigroup the average implied volatilities for a stock depreciation between 1% and 5% come from stock depreciations of 2.33% and 1.31%. However, the actual market data against which the Heston, SABR and Monte Carlo implied volatilities have been compared to has been retrieved after a drop in price of 3.80%.

10 - There is the assumption under this study that the prices estimated should represent the average of the bid-asks. However, is this the correct way of comparing them to the market data? For example, the SABR model often overestimates prices while the Heston and Monte Carlo models do the opposite. Would it then have been more appropriate, assuming the analyst is a price taker to use the SABR model just for bid estimations (ask calibrations if the analyst is a market maker) and the Heston and Monte Carlo models just

[241] Ibidem
[242] Ibidem
[243] http://quant.stackexchange.com/questions/5981/what-are-the-advantages-disadvantages-of-these-approaches-to-deal-with-volatilit
[244] Ibidem

for ask calibrations (bid estimations if the analyst is a market maker)?

11 - For small cap stocks there is insufficient options data to develop proper implied volatility surfaces, which means that the Heston and SABR can simply not be applied in this case to make a reasonable comparison. The Monte Carlo model however, can be used to estimate the single options prices.

12 - The Monte Carlo methodology cannot be used for American options.

10.4 Further applications of this study and general conclusions

What is also important for this study is how it can be used further for other applications. For example, we can state that by looking at Table 94, if the underlying has a movement in absolute value bigger than 5%, then the SABR model is the one to use for implied volatility surface and option prices calibrations if the company is a large-cap one. If instead the analyst is carrying out a study for mid-caps, the Heston model might be optimal for price estimations when the stock depreciates by more than 5%.

If we are analysing jumps in implied volatility for stocks other than those of this study, then the SABR model is the one that best captures such jumps for calls, while for puts it might be the case that both the Heston and the SABR models work pretty well.

If we are estimating volatilities of short-dated options, it would be more appropriate to adopt the SABR model. For long-dated options, both the Heston and the SABR model work equally well. On the other hand, for prices estimations, for short-dated options the Monte Carlo model can be quite precise too. For long-dated options, most of the times it would be best to use the SABR model.

For out-of-the-money options it is advised to use either the Heston or the SABR model whether the analyst is trying to estimate implied volatilities or option prices. For in-the-money options, instead, it is only optimal to use the SABR model.

The study could be further applied to estimate the implied volatility surfaces and prices of large Information Technology companies, such as Apple, and other big Oil & Gas firms, such as British Petroleum. There is strong evidence in this study from Financial Services stocks, which means that the study could be further applied to other banks such as Bank of America and J.P. Morgan. In this case it would be optimal to use the SABR model for both implied volatilities and prices estimations. For mid-cap banks, Zions Bancorporation's results make the analyst shift towards using the Heston model for option prices calibration.

With respect to news and headlines, we have seen that catalysts for comparable banks might have a major impact in the change in levels of the implied volatility surface and in the changes in prices of both calls and puts. This can be further extended to other bank stocks, whether large-cap, mid-cap or small-cap and whether global, commercial or investment banks.

However, it must be noted that with respect to option price calibrations, all of the three models seem to be more precise for banks than for Information Technology or Oil & Gas. Indeed, it suffices to see the price differentials to see that the absolute value of such differentials is much smaller for options of banks than otherwise. It must also be underlined that this is mainly true because the underlying sock prices are much smaller for bank stocks than for Google and Exxon. If the analyst wants to invest $X in a company options, then he could also look at the percentage proportional discrepancy between option prices given by the market and by the models.

So, we can also conclude and keep in mind that the SABR model is equally precise independently of the stock sector for implied volatilities and equally precise independently of the stock sector for option price estimations as long as the stock belongs to a large-cap company. Furthermore is is more precise to use for large-cap bank companies rather the mid-cap ones. For the Heston model, more appropriate results can be used when analysing banks, and not when carrying out studies on Information Technology or Oil & Gas firms. The Heston model is more appropriate for mid-cap banks, such as Zions Bancorporation, for both implied volatility surface estimations and option price calibrations. Lastly, the Monte Carlo model should be used exclusively for option pricing and it can be precise for both large-cap and mid-cap companies. Such model is also more precise with respect to banks when the bank has only one specific specialisation, such as Goldman Sachs in Investment Banking or Zions Bancorporation in Commercial Banking.

Therefore, the sector choice does change the performance of the specific model and the "best" model does change based on the sector, on the market capitalisation and on the specialty the firm is involved in.

With respect to other asset classes, it is more tricky to apply the results of this study to them since, as we have seen in the Theoretical Section of this research paper, Equity options present mainly skews and FX ones are characterised by smiles. Commodities could have either. A reasonable approach could be to use the results of the Financial Services sector to estimate FX option prices, as banks's trading desks also work with FX instruments and currencies. The same can be said for Fixed-Income options. For commodities, or at least for Oil, Exxon Mobil's results of this study could be used as plausible ones to estimate Oil option implied volatilities and prices.

Other aspects of this study that could further analysed are that Oil & Gas companies are a lot dependent on Oil prices, as well as banks' financial performances are tied to regulation and Information Technology firms are dependent on the number of users per application. We could therefore carry out regression analyses on whether such aspects of the various industries have an effect on which model is the best one to use for calibrations of implied volatility surfaces and of option prices.

So as we have seen in this section of the research paper, it does matter whether the company is a mid-cap, large-cap or very large-cap, whether the stock price appreciates or depreciates by X%, whether the company is in the Financial Services industry or not, whether there are catalysts coming on for the specific trading day, and whether we are analysing a call or put option.

Lastly, the Conclusions Section of this paper has been trying to explain why this study has had the results that it has had. Reasons vary from what stated in the previous paragraph to other factors, such as possible changes in interest rates, dividends, time passing by, volumes and open interest being relatively low or high for specific options being traded, wide or narrow bid-ask spreads, transaction costs, taxes, funding liquidity, market liquidity, the stock market the underlying is being traded in and the asset class being analysed.

The best improvement that can be done for this study is to carry out various regression analyses to see which os all these factors are statistically significant for various confidence intervals with respect to the performance of the models.

Moreover, there could be a comparison of the Heston, SABR and Monte Carlo models to other more sophisticated ones, such as the Heston-Nandi model and the SVJJ construction methodology. By using models such as the SVJJ we can allow for jumps in the stock price and in volatility to exist. We would therefore be able to see whether such models are effectively more precise than more standard ones for stock jumps, or also for instances where the stock price moves by less than 5%.

However, the main conclusion of the empirical studies carried out in this research paper is that if the analyst has a limited amount of time and if he can use only one of the three models analysed in this research paper to carry out studies for stocks of various sectors, he/she should then use the SABR model.

References

[1] A. Medvedev and O. Scaillet. Approximation and calibration of short-term implied volatil- ities under jump-diffusion stochastic volatility. http://www.hec.unige.ch/www/hec/m1/ RecherchePublications/CahiersRecherche/2006/simpleText/0/content files/file8/2006. 01.01.01.01.01.01.01.01.01.01.01.01.pdf, 2006.

[2] B. Dumas, J. Fleming, and R.E: Whaley. Implied volatility functions: Empirical tests. *The Journal of Finance. (1998)*

[3] C. Humescu. *Implied volatility surface: construction methodologies and characteristics. (2011).*

[4] Carr and Wu. *A new simple approach for constructing implied volatility surfaces. (2011)*

[5] D. Brecher. *Pushing the limits of local volatility in option pricing. (2006)*

[6] D. Kohlberg. *The interest volatility surface. (2011)*

[7] D.C. Brody, I.R.C. Buckley, and I.C. Constantinou. *Option price calibration from Renyi entropy. (2007)*

[8] E. Nilsson. *Calibration of the Volatility Surface. 2008*

[9] http://arxiv.org/pdf/1107.1834.pdf

[10] http://blogs.barrons.com/techtraderdaily/2016/04/26/google-slips-ftc-probe-possible-android-dominance-abuse-says-wsj/?mod=yahoobarrons&ru=yahoo

[11] http://cermics.enpc.fr/cnf/Guillaume.pdf

[12] http://edoc.hu-berlin.de/series/sfb-649-papers/2008-44/PDF/44.pdf

[13] http://faculty.baruch.cuny.edu/jgatheral/impliedvolatilitysurface.pdf

[14] http://faculty.baruch.cuny.edu/lwu/9797/Lec8.pdf

[15] http://finance.yahoo.com/news/bank-america-profit-drops-18-112937514.html

[16] http://finance.yahoo.com/news/citi-revises-executive-pay-plan-205808263.html

[17] http://finance.yahoo.com/news/goldman-sachs-profit-slumps-fourth-114645384.html

[18] http://finance.yahoo.com/news/stocks-on-edge-as-fed-commentary–jobs-report-looms-191358208.html

[19] http://finance.yahoo.com/news/stocks-retreat-as-oil–dollar-push-lowe-191735867.html

[20] http://finance.yahoo.com/q?s=C

[21] http://finance.yahoo.com/q?s=GOOG

[22] http://finance.yahoo.com/q?s=GS

[23] http://finance.yahoo.com/q?s=XOM

[24] http://finance.yahoo.com/q?s=ZION

[25] http://finance.yahoo.com/q/op?s=C+Options

[26] http://finance.yahoo.com/q/op?s=GOOG+Options

[27] http://finance.yahoo.com/q/op?s=GS+Options

[28] http://finance.yahoo.com/q/op?s=XOM+Options

[29] http://finance.yahoo.com/q/op?s=ZION+Options

[30] http://finance.yahoo.com/q/pr?s=GOOG+Profile

[31] http://finance.yahoo.com/q/pr?s=GS+Profile

[32] http://finance.yahoo.com/q/pr?s=XOM+Profile

[33] http://it.mathworks.com/matlabcentral/fileexchange/29446-heston-model-calibration-and-simulation

[34] http://kodu.ut.ee/ spartak/papers/sabr.pdf

[35] http://pure.au.dk/portal-asb-student/files/8505/222384.pdf

[36] http://quant.stackexchange.com/questions/5981/what-are-the-advantages-disadvantages-of-these-approaches-to-deal-with-volatilit

[37] http://www-2.rotman.utoronto.ca/ hull/downloadablepublications/DHSPaperdraft7.pdf

[38] http://www-2.rotman.utoronto.ca/ hull/downloadablepublications/DHSPaperdraft7.pdf

[39] http://www-math.unice.fr/ diener/nicemathfi03/Bruno Dupire vol presentation 1302c.ppt.

[40] http://www.bloomberg.com/news/videos/2016-04-15/goldman-said-to-seek-deep-cost-cuts?cmpid=yhoo.headline

[41] http://www.cmap.polytechnique.fr/ rama/teaching/ea2003/gatheral.pdf

[42] http://www.columbia.edu/ mh2078/BlackScholesCtsTime.pdf

[43] http://www.fam.tuwien.ac.at/ sgerhold/pub files/sem12/s sibetz nowak.pdf

[44] http://www.forbes.com/companies/citigroup/

[45] http://www.iasonltd.com/wp-content/uploads/2013/02/2b.pdf

[46] http://www.investopedia.com/terms/i/iv.asp

[47] http://www.investopedia.com/terms/v/volatility.asp

[48] http://www.investors.com/news/technology/citi-lowers-amazon-nflx-google-price-targets-on-stock-compensation/?ven=YahooCP&src=AURLLED&ven=yahoo

[49] http://www.marketwatch.com/story/citigroups-stock-drops-after-analyst-downgrade-2016-04-18?siteid=yhoof2

[50] http://www.marketwatch.com/story/goldman-cut-pay-for-executives-in-2015-2016-04-08?siteid=yhoof2

[51] http://www.marketwatch.com/story/nyse-suspends-trading-of-amazon-alphabet-6-other-securities-2016-04-01?siteid=yhoof2

[52] http://www.mathworks.com/matlabcentral/fileexchange/23316-volatility-surface/content/VolSurface.m

[53] http://www.performancetrading.it/Documents/KscComparison/Ksc Local.htm

[54] http://www.siliconbeat.com/2016/04/04/google-beats-back-patent-troll-saves-85-million/

[55] http://www.spvixviews.com/wp-content/uploads/2012/02/Exhibit-13-SP-500-Implied-Volatility-vs.-Subsequent-Realized-Volatility.jpg

[56] http://www2.math.su.se/matstat/reports/seriec/2011/rep7/report.pdf

[57] http://www2.math.uu.se/ maciej/courses/PDE for Finance/DF2012.pdf

[58] https://en.wikipedia.org/wiki/Local volatility

[59] https://en.wikipedia.org/wiki/Monte Carlo methods for option pricing

[60] https://en.wikipedia.org/wiki/Volatility smile

[61] https://en.wikipedia.org/wiki/Zions Bancorporation

[62] https://people.maths.ox.ac.uk/reisinge/Students/volaNotes.pdf

[63] https://www.researchgate.net/publication/257200550 Static and dynamic SABR stochastic volatility models Calibration and option pricing using GPUs

[64] https://www.tradeking.com/education/options/what-is-implied-volatility

[65] J. C. Hull. *Options, Futures, and other Derivatives - Eighth Edition. (2012)*

[66] J. Gatheral. *The volatility surface: a practitioner's guide. (2006)*

[67] J. Maruhn. *On-the-fly bid/ask-vol fitting with applications in model calibration. (2010)*

[68] L. Marroni, I. Perdomo. *Pricing and Hedging Financial Derivatives - A Guide for Practitioners.* (English)

[69] M. Benko, M.R. Fengler, W.K. Hardle, and M.Kopa. *On extracting information implied in options. (2007).*

[70] M. Haug *Black-Scholes and the Volatility Surface.*

[71] M. Laurini. *Imposing no-arbitrage conditions in implied volatility surfaces using constrained smoothing splines. (2007)*

[72] M. Poletti Laurini *Imposing no-arbitrage conditions in implied volatility surfaces using constrained smoothing splines. (2007).*

[73] M. R. Fengler *Arbitrage-Free Smoothing of the Implied Volatility Surface. (2005).*

[74] M.R. Fengler. *Option data and modeling BSM implied volatility. (2010)*

[75] M.V. Deryabin. *Implied volatility surface reconstruction for energy markets: spot price modeling versus surface parametrization. (2011)*

[76] N. Kahale. *An arbitrage-free interpolation of volatilities. (2004).*

[77] O. Bossard: Monte Carlo and Black-Scholes Excel VBAs

[78] S. Borovkova and F. J. Permana. *Implied volatility in oil markets. Computational Statistics and Data Analysis. (2009)*

[79] www.prmia.org/sites/default/files/legacy/Chapter.../Greg%20Orosi.ppt

[80] Y. Osajima. *The Asymptotic Expansion Formula of Implied Volatility for Dynamic SABR Model and FX Hybrid Model. (2007)*

[81] Y. Wang, H. Yin and L.Qi. *No-arbitrage interpolation of the option price function and its reformulation. (2004)*

Appendix A

The following code shows the calibration for the SABR parameters (α, β, ρ, ν) obtained form averaging the implied volatilities after the underlying price movement has been in the same range twice (in this case, after the moment after the underlying has moved for the second time between -5% and -1%) for a capital depreciation of 3.80% registered on April 7^{th} 2016, with the underlying price reaching the $40.27 figure:

```
global impvol; global strike; global T; global F0; global r;

x0=[.5,.5,.5,.05,.5];
lb = [0, 0, 0, 0, -.9];
ub = [1, 100, 1, .5, .9];
M=10000;
F0=40.27; r=0.0025;

x = xlsread( 'C.xlsx', 'Data Call -5 to -1 3');

%Calibration of the SABR parameters

Settle = '07-Apr-2016';
ExerciseDate = '20-May-2016';
ExerciseDate2 = '16-Sep-2016';
ExerciseDate3 = '20-Jan-2017';
ExerciseDate4 = '19-Jan-2018';

MarketStrikes = [35 40 45 50 55]'/100;
MarketVolatilities =x(:,2);
MarketVolatilities2 =x(:,3);
MarketVolatilities3 =x(:,4);
MarketVolatilities4 =x(:,5);

CurrentForwardValue = F0/100;

%FIRST MATURITY - Method 1

Beta1 = 0.5;

% Calibrate Alpha, Rho, and Nu
objFun = @(X) MarketVolatilities - ...
  blackvolbysabr(X(1), Beta1, X(2), X(3), Settle, ...
  ExerciseDate, CurrentForwardValue, MarketStrikes);

X = lsqnonlin(objFun, [0.5 0 0.5], [0 -1 0], [Inf 1 Inf]);

Alpha1 = X(1);
Rho1 = X(2);
Nu1 = X(3);

% Display calibrated parameters
C = Alpha1 Beta1 Rho1 Nu1;
CalibratedPrameters = cell2table(C,...
  'VariableNames','Alpha' 'Beta' 'Rho' 'Nu',...
  'RowNames','Maturity 1')

%SECOND MATURITY - Method 2
```

```
Beta2 = 0.5;

% Calibrate Alpha, Rho, and Nu
objFun = @(X) MarketVolatilities2 - ...
   blackvolbysabr(X(1), Beta2, X(2), X(3), Settle, ...
   ExerciseDate2, CurrentForwardValue, MarketStrikes);

X = lsqnonlin(objFun, [0.5 0 0.5], [0 -1 0], [Inf 1 Inf]);

Alpha2 = X(1);
Rho2 = X(2);
Nu2 = X(3);

% Display calibrated parameters
C = Alpha2 Beta2 Rho2 Nu2;
CalibratedPrameters = cell2table(C,...
   'VariableNames','Alpha' 'Beta' 'Rho' 'Nu',...
   'RowNames','Maturity 2')

%THIRD MATURITY - Method 3

Beta3 = 0.5;

% Calibrate Alpha, Rho, and Nu
objFun = @(X) MarketVolatilities3 - ...
   blackvolbysabr(X(1), Beta3, X(2), X(3), Settle, ...
   ExerciseDate3, CurrentForwardValue, MarketStrikes);

X = lsqnonlin(objFun, [0.5 0 0.5], [0 -1 0], [Inf 1 Inf]);

Alpha3 = X(1);
Rho3 = X(2);
Nu3 = X(3);

% Display calibrated parameters
C = Alpha3 Beta3 Rho3 Nu3;
CalibratedPrameters = cell2table(C,...
   'VariableNames','Alpha' 'Beta' 'Rho' 'Nu',...
   'RowNames','Maturity 3')

%FOURTH MATURITY - Method 4

Beta4 = 0.5;

% Calibrate Alpha, Rho, and Nu
objFun = @(X) MarketVolatilities4 - ...
   blackvolbysabr(X(1), Beta4, X(2), X(3), Settle, ...
   ExerciseDate4, CurrentForwardValue, MarketStrikes);

X = lsqnonlin(objFun, [0.5 0 0.5], [0 -1 0], [Inf 1 Inf]);

Alpha4 = X(1);
Rho4 = X(2);
Nu4 = X(3);

% Display calibrated parameters
```

C = Alpha4 Beta4 Rho4 Nu4;
CalibratedPrameters = cell2table(C,...
'VariableNames','Alpha' 'Beta' 'Rho' 'Nu',...
'RowNames','Maturity 4')

The following figure shows the various Excel worksheets to make the SABR parameters calibration work:

Figure 301: Data Call -5 to -1 1

	A	B	C	D	E
1	35	43.85%	45.31%	35.11%	37.95%
2	40	31.91%	30.45%	30.21%	35.69%
3	45	26.66%	26.47%	27.34%	30.92%
4	50	25.68%	23.78%	25.22%	27.24%
5	55	29.69%	22.41%	23.15%	26.20%

Figure 302: Data Call -5 to -1 2

	A	B	C	D	E
1	35	44.82%	44.29%	37.23%	34.72%
2	40	31.67%	29.68%	30.42%	31.53%
3	45	27.64%	26.36%	27.31%	28.27%
4	50	29.30%	24.12%	24.93%	26.47%
5	55	37.50%	23.00%	23.24%	25.29%

Figure 303: Data Call -5 to -1 3

	A	B	C	D	E
1	35	44.34%	44.80%	36.17%	36.34%
2	40	31.79%	30.07%	30.32%	33.61%
3	45	27.15%	26.42%	27.33%	29.60%
4	50	27.49%	23.95%	25.08%	26.86%
5	55	33.60%	22.71%	23.20%	25.75%

Appendix B

The following appendix shows the code to the Monte Carlo simulation to find both option prices and implied volatilities (through the use of the Black and Scholes formula), carried out on excel:

```
Sub MonteCarlo()

nsim = Cells(3, 7)

p1 = 0
p2 = 0
p3 = 0
p4 = 0
p5 = 0
p6 = 0
p7 = 0
p8 = 0
p9 = 0
p10 = 0
p11 = 0
p12 = 0
p13 = 0
p14 = 0
p15 = 0
p16 = 0
p17 = 0
p18 = 0
p19 = 0
p20 = 0
p21 = 0
p22 = 0
p23 = 0
p24 = 0

p1_squared = 0
p2_squared = 0
p3_squared = 0
p4_squared = 0
p5_squared = 0
p6_squared = 0
p7_squared = 0
p8_squared = 0
p9_squared = 0
p10_squared = 0
p11_squared = 0
p12_squared = 0
p13_squared = 0
p14_squared = 0
p15_squared = 0
p16_squared = 0
p17_squared = 0
p18_squared = 0
p19_squared = 0
p20_squared = 0
p21_squared = 0
p22_squared = 0
p23_squared = 0
```

$p24\_squared = 0$

Cells.Select
ActiveSheet.Calculate
Application.ScreenUpdating = False

For i = 1 To nsim
   Range("F13:Z42").Select
   ActiveSheet.Calculate

$new\_p1 = Cells(21, 21)$
$new\_p2 = Cells(22, 21)$
$new\_p3 = Cells(23, 21)$
$new\_p4 = Cells(24, 21)$
$new\_p5 = Cells(21, 22)$
$new\_p6 = Cells(22, 22)$
$new\_p7 = Cells(23, 22)$
$new\_p8 = Cells(24, 22)$
$new\_p9 = Cells(21, 23)$
$new\_p10 = Cells(22, 23)$
$new\_p11 = Cells(23, 23)$
$new\_p12 = Cells(24, 23)$
$new\_p13 = Cells(21, 24)$
$new\_p14 = Cells(22, 24)$
$new\_p15 = Cells(23, 24)$
$new\_p16 = Cells(24, 24)$
$new\_p17 = Cells(21, 25)$
$new\_p18 = Cells(22, 25)$
$new\_p19 = Cells(23, 25)$
$new\_p20 = Cells(24, 25)$
$new\_p21 = Cells(21, 26)$
$new\_p22 = Cells(22, 26)$
$new\_p23 = Cells(23, 26)$
$new\_p24 = Cells(24, 26)$

$p1 = p1 + new\_p1$
$p2 = p2 + new\_p2$
$p3 = p3 + new\_p3$
$p4 = p4 + new\_p4$
$p5 = p5 + new\_p5$
$p6 = p6 + new\_p6$
$p7 = p7 + new\_p7$
$p8 = p8 + new\_p8$
$p9 = p9 + new\_p9$
$p10 = p10 + new\_p10$
$p11 = p11 + new\_p11$
$p12 = p12 + new\_p12$
$p13 = p13 + new\_p13$
$p14 = p14 + new\_p14$
$p15 = p15 + new\_p15$
$p16 = p16 + new\_p16$
$p17 = p17 + new\_p17$
$p18 = p18 + new\_p18$
$p19 = p19 + new\_p19$
$p20 = p20 + new\_p20$
$p21 = p21 + new\_p21$

$p22 = p22 + new\_p22$
$p23 = p23 + new\_p23$
$p24 = p24 + new\_p24$

$p1\_squared = p1\_squared + new_p1 * new_p1$
$p2\_squared = p2\_squared + new_p2 * new_p2$
$p3\_squared = p3\_squared + new_p3 * new_p3$
$p4\_squared = p4\_squared + new_p4 * new_p4$
$p5\_squared = p5\_squared + new_p5 * new_p5$
$p6\_squared = p6\_squared + new_p6 * new_p6$
$p7\_squared = p7\_squared + new_p7 * new_p7$
$p8\_squared = p8\_squared + new_p8 * new_p8$
$p9\_squared = p9\_squared + new_p9 * new_p9$
$p10\_squared = p10\_squared + new_p10 * new_p10$
$p11\_squared = p11\_squared + new_p11 * new_p11$
$p12\_squared = p12\_squared + new_p12 * new_p12$
$p13\_squared = p13\_squared + new_p13 * new_p13$
$p14\_squared = p14\_squared + new_p14 * new_p14$
$p15\_squared = p15\_squared + new_p15 * new_p15$
$p16\_squared = p16\_squared + new_p16 * new_p16$
$p17\_squared = p17\_squared + new_p17 * new_p17$
$p18\_squared = p18\_squared + new_p18 * new_p18$
$p19\_squared = p19\_squared + new_p19 * new_p19$
$p20\_squared = p20\_squared + new_p20 * new_p20$
$p21\_squared = p21\_squared + new_p21 * new_p21$
$p22\_squared = p22\_squared + new_p22 * new_p22$
$p23\_squared = p23\_squared + new_p23 * new_p23$
$p24\_squared = p24\_squared + new_p24 * new_p24$

Cells(4, 7) = i
Next i

Application.ScreenUpdating = True

$std\_dev1 = Sqr((nsim * p1\_squared - p1 * p1)/(nsim * (nsim - 1)))$
$std\_dev2 = Sqr((nsim * p2\_squared - p2 * p2)/(nsim * (nsim - 1)))$
$std\_dev3 = Sqr((nsim * p3\_squared - p3 * p3)/(nsim * (nsim - 1)))$
$std\_dev4 = Sqr((nsim * p4\_squared - p4 * p4)/(nsim * (nsim - 1)))$
$std\_dev5 = Sqr((nsim * p5\_squared - p5 * p5)/(nsim * (nsim - 1)))$
$std\_dev6 = Sqr((nsim * p6\_squared - p6 * p6)/(nsim * (nsim - 1)))$
$std\_dev7 = Sqr((nsim * p7\_squared - p7 * p7)/(nsim * (nsim - 1)))$
$std\_dev8 = Sqr((nsim * p8\_squared - p8 * p8)/(nsim * (nsim - 1)))$
$std\_dev9 = Sqr((nsim * p9\_squared - p9 * p9)/(nsim * (nsim - 1)))$
$std\_dev10 = Sqr((nsim * p10\_squared - p10 * p10)/(nsim * (nsim - 1)))$
$std\_dev11 = Sqr((nsim * p11\_squared - p11 * p11)/(nsim * (nsim - 1)))$
$std\_dev12 = Sqr((nsim * p12\_squared - p12 * p12)/(nsim * (nsim - 1)))$
$std\_dev13 = Sqr((nsim * p13\_squared - p13 * p13)/(nsim * (nsim - 1)))$
$std\_dev14 = Sqr((nsim * p14\_squared - p14 * p14)/(nsim * (nsim - 1)))$
$std\_dev15 = Sqr((nsim * p15\_squared - p15 * p15)/(nsim * (nsim - 1)))$
$std\_dev16 = Sqr((nsim * p16\_squared - p16 * p16)/(nsim * (nsim - 1)))$
$std\_dev17 = Sqr((nsim * p17\_squared - p17 * p17)/(nsim * (nsim - 1)))$
$std\_dev18 = Sqr((nsim * p18\_squared - p18 * p18)/(nsim * (nsim - 1)))$
$std\_dev19 = Sqr((nsim * p19\_squared - p19 * p19)/(nsim * (nsim - 1)))$
$std\_dev20 = Sqr((nsim * p20\_squared - p20 * p20)/(nsim * (nsim - 1)))$
$std\_dev21 = Sqr((nsim * p21\_squared - p21 * p21)/(nsim * (nsim - 1)))$
$std\_dev22 = Sqr((nsim * p22\_squared - p22 * p22)/(nsim * (nsim - 1)))$

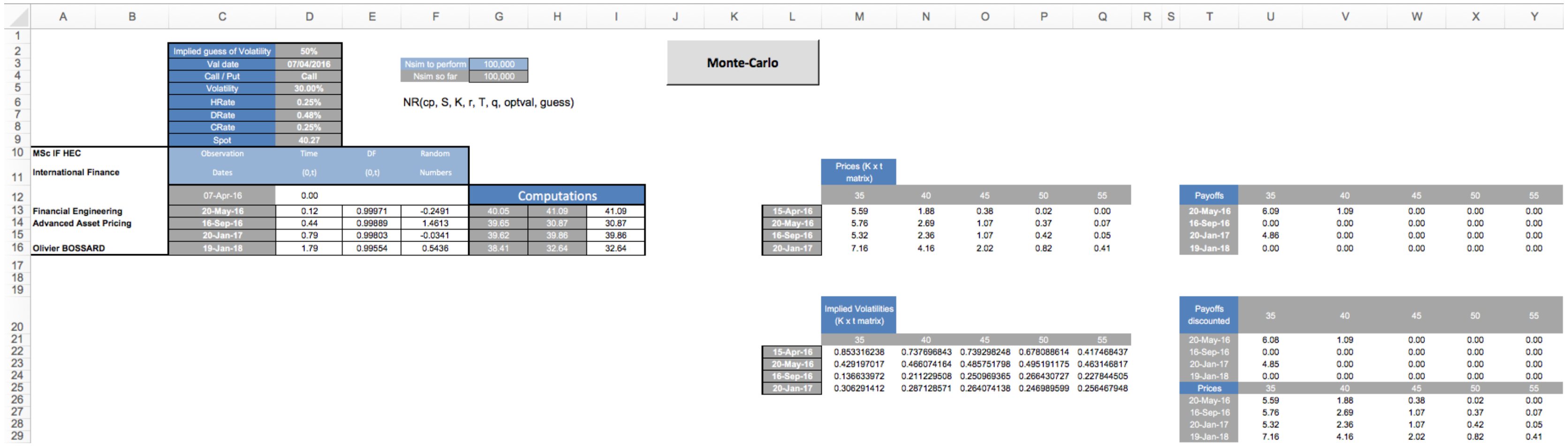
$std\_dev23 = Sqr((nsim * p23\_squared - p23 * p23)/(nsim * (nsim - 1)))$
$std\_dev24 = Sqr((nsim * p24\_squared - p24 * p24)/(nsim * (nsim - 1)))$

Cells(26, 21) = p1 / nsim
Cells(27, 21) = p2 / nsim
Cells(28, 21) = p3 / nsim
Cells(29, 21) = p4 / nsim
Cells(26, 22) = p5 / nsim
Cells(27, 22) = p6 / nsim
Cells(28, 22) = p7 / nsim
Cells(29, 22) = p8 / nsim
Cells(26, 23) = p9 / nsim
Cells(27, 23) = p10 / nsim
Cells(28, 23) = p11 / nsim
Cells(29, 23) = p12 / nsim
Cells(26, 24) = p13 / nsim
Cells(27, 24) = p14 / nsim
Cells(28, 24) = p15 / nsim
Cells(29, 24) = p16 / nsim
Cells(26, 25) = p17 / nsim
Cells(27, 25) = p18 / nsim
Cells(28, 25) = p19 / nsim
Cells(29, 25) = p20 / nsim
Cells(26, 26) = p21 / nsim
Cells(27, 26) = p22 / nsim
Cells(28, 26) = p23 / nsim
Cells(29, 26) = p24 / nsim

End Sub

The following figure shows the respective Excel worksheet to make the Monte Carlo method work, with all the given parameters:

Figure 304: Monte Carlo worksheet - Citigroup



Implied guess of Volatility	50%
Val date	07/04/2016
Call / Put	Call
Volatility	30.00%
HRate	0.25%
DRate	0.48%
CRate	0.25%
Spot	40.27

Nsim to perform	100,000
Nsim so far	100,000

NR(cp, S, K, r, T, q, optval, guess)

Monte-Carlo

MSc IF HEC

International Finance

Financial Engineering

Advanced Asset Pricing

Olivier BOSSARD

Observation Dates	Time (0,t)	DF (0,t)	Random Numbers	Computations		
07-Apr-16	0.00					
20-May-16	0.12	0.99971	-0.2491	40.05	41.09	41.09
16-Sep-16	0.44	0.99889	1.4613	39.65	30.87	30.87
20-Jan-17	0.79	0.99803	-0.0341	39.62	39.86	39.86
19-Jan-18	1.79	0.99554	0.5436	38.41	32.64	32.64

Prices (K x t matrix)	35	40	45	50	55
15-Apr-16	5.59	1.88	0.38	0.02	0.00
20-May-16	5.76	2.69	1.07	0.37	0.07
16-Sep-16	5.32	2.36	1.07	0.42	0.05
20-Jan-17	7.16	4.16	2.02	0.82	0.41

Implied Volatilities (K x t matrix)	35	40	45	50	55
15-Apr-16	0.853316238	0.737696843	0.739298248	0.678088614	0.417468437
20-May-16	0.429197017	0.466074164	0.485751798	0.495191175	0.463148817
16-Sep-16	0.136633972	0.211229508	0.250969365	0.266430727	0.227844505
20-Jan-17	0.306291412	0.287128571	0.264074138	0.246989599	0.256467948

Payoffs	35	40	45	50	55
20-May-16	6.09	1.09	0.00	0.00	0.00
16-Sep-16	0.00	0.00	0.00	0.00	0.00
20-Jan-17	4.86	0.00	0.00	0.00	0.00
19-Jan-18	0.00	0.00	0.00	0.00	0.00

Payoffs discounted	35	40	45	50	55
20-May-16	6.08	1.09	0.00	0.00	0.00
16-Sep-16	0.00	0.00	0.00	0.00	0.00
20-Jan-17	4.85	0.00	0.00	0.00	0.00
19-Jan-18	0.00	0.00	0.00	0.00	0.00
Prices	35	40	45	50	55
20-May-16	5.59	1.88	0.38	0.02	0.00
16-Sep-16	5.76	2.69	1.07	0.37	0.07
20-Jan-17	5.32	2.36	1.07	0.42	0.05
19-Jan-18	7.16	4.16	2.02	0.82	0.41

Appendix C

The following appendix shows the MATLAB 2016 code used to find the Heston and SABR implied volatilities, Heston and SABR option prices and plot them as well as the Monte Carlo implied volatilities and prices and market implied volatilities and prices found on the relevant Yahoo! Finance website (example showing the calculation of Heston and SABR call implied volatilities and prices for Citigroup in 1, 5, 9 and 21 months, strikes (per share) at \$35, \$40, \$45, \$50, \$55; with spot price equal to \$40.27 per share (closing of business day April 7^{th} 2016 for a daily capital depreciation of 3.80%)):

```
global impvol; global strike; global T; global F0; global r;

x0=[.5,.5,.5,.05,.5];
lb = [0, 0, 0, 0, -.9];
ub = [1, 100, 1, .5, .9];
M=10000;
F0=40.27; r=0.0025;

x = xlsread( 'C.xlsx', 'Heston and SABR Call -5 to -1');

T=x(:,5)/365;
T2=x(:,10)/365;
T3=x(:,15)/365;
T4=x(:,20)/365;
strike=x(:,1);
strike2=x(:,6);
strike3=x(:,11);
strike4=x(:,16);
impvol=x(:,4);
impvol2=x(:,9);
impvol3=x(:,14);
impvol4=x(:,19);
priceoption=x(:,3);
priceoption2=x(:,8);
priceoption3=x(:,13);
priceoption4=x(:,18);

%Optimization
y = lsqnonlin(@costf2,x0,lb,ub);

for k=1:length(T);
%Initial asset price
shes(1)=F0;
%Number of Time Steps,time step size
N=round(T(k)/(1/360));dt=T(k)/N;

%Heston Parameters
vhes(1)=y(1); kappa=y(2); theta=y(3); vsigma=y(4);rho=y(5); simPath=0;

%Simulation of Heston Model
for i = 1:M
for j=1:N
r1 = randn;
r2 = rho*r1+sqrt(1-rho*rho)*randn;
shes(j+1)=shes(j)*exp((-0.5*vhes(j))*dt+sqrt(vhes(j))*sqrt(dt)*r1);
vhes(j+1)=vhes(j)*exp...
  (((kappa*(theta - vhes(j))-0.5*vsigma*vsigma)*dt)/vhes(j) + vsigma*(1/sqrt(vhes(j)))*sqrt(dt)*r2);
end
```

```
end
modhes(k)=HestonCall(shes(1),strike(k),r,T(k),vhes(1),kappa,theta,vsigma,rho,0);
hesimpvol(k)=blkimpv(shes(1), strike(k), r, T(k), modhes(k));
end

%Output optimized Parameters
y;

for k=1:length(T2);
%Initial asset price
shes(1)=F0;
%Number of Time Steps,time step size
N=round(T2(k)/(1/360));dt=T2(k)/N;

%Heston Parameters
vhes(1)=y(1); kappa=y(2); theta=y(3); vsigma=y(4);rho=y(5); simPath=0;

%Simulation of Heston Model
for i = 1:M
for j=1:N
r1 = randn;
r2 = rho*r1+sqrt(1-rho*rho)*randn;
shes(j+1)=shes(j)*exp((-0.5*vhes(j))*dt+sqrt(vhes(j))*sqrt(dt)*r1);
vhes(j+1)=vhes(j)*exp...
   (((kappa*(theta - vhes(j))-0.5*vsigma*vsigma)*dt)/vhes(j) + vsigma*(1/sqrt(vhes(j)))*sqrt(dt)*r2);
end
end
modhes2(k)=HestonCall(shes(1),strike2(k),r,T2(k),vhes(1),kappa,theta,vsigma,rho,0);
hesimpvol2(k)=blkimpv(shes(1), strike2(k), r, T2(k), modhes2(k));
end

%Output optimized Parameters
y;

for k=1:length(T3);
%Initial asset price
shes(1)=F0;
%Number of Time Steps,time step size
N=round(T3(k)/(1/360));dt=T3(k)/N;

%Heston Parameters
vhes(1)=y(1); kappa=y(2); theta=y(3); vsigma=y(4);rho=y(5); simPath=0;

%Simulation of Heston Model
for i = 1:M
for j=1:N
r1 = randn;
r2 = rho*r1+sqrt(1-rho*rho)*randn;
shes(j+1)=shes(j)*exp((-0.5*vhes(j))*dt+sqrt(vhes(j))*sqrt(dt)*r1);
vhes(j+1)=vhes(j)*exp...
   (((kappa*(theta - vhes(j))-0.5*vsigma*vsigma)*dt)/vhes(j) + vsigma*(1/sqrt(vhes(j)))*sqrt(dt)*r2);
end
end
modhes3(k)=HestonCall(shes(1),strike3(k),r,T3(k),vhes(1),kappa,theta,vsigma,rho,0);
hesimpvol3(k)=blkimpv(shes(1), strike3(k), r, T3(k), modhes3(k));
end
```

```
%Output optimized Parameters
y;

for k=1:length(T4);
%Initial asset price
shes(1)=F0;
%Number of Time Steps,time step size
N=round(T4(k)/(1/360));dt=T4(k)/N;

%Heston Parameters
vhes(1)=y(1); kappa=y(2); theta=y(3); vsigma=y(4);rho=y(5); simPath=0;

%Simulation of Heston Model
for i = 1:M
for j=1:N
r1 = randn;
r2 = rho*r1+sqrt(1-rho*rho)*randn;
shes(j+1)=shes(j)*exp((-0.5*vhes(j))*dt+sqrt(vhes(j))*sqrt(dt)*r1);
vhes(j+1)=vhes(j)*exp...
   (((kappa*(theta - vhes(j))-0.5*vsigma*vsigma)*dt)/vhes(j) + vsigma*(1/sqrt(vhes(j)))*sqrt(dt)*r2);
end
end
modhes4(k)=HestonCall(shes(1),strike4(k),r,T4(k),vhes(1),kappa,theta,vsigma,rho,0);
hesimpvol4(k)=blkimpv(shes(1), strike4(k), r, T4(k), modhes4(k));
end

%Output optimized Parameters
y;

pricedata1= [modhes'-priceoption, modhes2'-priceoption2, modhes3'-priceoption3, modhes4'-priceoption4];

voldata=[hesimpvol', hesimpvol2', hesimpvol3', hesimpvol4'];

t = [1.5, 5.5, 9.5, 21.5];
w = [F0/35, F0/40, F0/45, F0/50, F0/55];

surf(t,w,voldata)

xlabel('Months to maturity', 'FontWeight', 'bold')
ylabel('Moneyness', 'FontWeight', 'bold')
zlabel('Heston Implied Volatility', 'FontWeight', 'bold')
colorbar

figure; surf(t,w,pricedata1)

xlabel('Months to maturity', 'FontWeight', 'bold')
ylabel('Moneyness', 'FontWeight', 'bold')
zlabel('Heston Price - Market Price', 'FontWeight', 'bold')
colorbar

%SABR Model

Settle = '07-Apr-2016';
ExerciseDate = '20-May-2016';
ExerciseDate2 = '16-Sep-2016';
```

```
ExerciseDate3 = '20-Jan-2017';
ExerciseDate4 = '19-Jan-2018';

MarketStrikes = [35 40 45 50 55]'/100;
MarketVolatilities =x(:,4);
MarketVolatilities2 =x(:,9);
MarketVolatilities3 =x(:,14);
MarketVolatilities4 =x(:,19);

CurrentForwardValue = F0/100;

PlottingStrikes = [35 40 45 50 55]'/100;

% Compute volatilities for model calibrated by Method 1
ComputedVols = blackvolbysabr(0.20, 0.5, -0.59, 2.41, Settle, ...
  ExerciseDate, CurrentForwardValue, PlottingStrikes);

% Compute volatilities for model calibrated by Method 2
ComputedVols2 = blackvolbysabr(0.21, 0.5, -0.76, 1.74, Settle, ...
  ExerciseDate2, CurrentForwardValue, PlottingStrikes);

% Compute volatilities for model calibrated by Method 3
ComputedVols3 = blackvolbysabr(0.19, 0.5, -0.64, 0.84, Settle, ...
  ExerciseDate3, CurrentForwardValue, PlottingStrikes);

% Compute volatilities for model calibrated by Method 4
ComputedVols4 = blackvolbysabr(0.21, 0.5, -0.65, 0.61, Settle, ...
  ExerciseDate4, CurrentForwardValue, PlottingStrikes);

t = [1.5, 5.5, 9.5, 21.5];
z = [ComputedVols, ComputedVols2, ComputedVols3, ComputedVols4];

figure; surf(t,w,z)
xlabel('Months to maturity', 'FontWeight', 'bold')
ylabel('Moneyness', 'FontWeight', 'bold')
zlabel('SABR Implied Volatility', 'FontWeight', 'bold')
colorbar

BlackScholesPrice=blsprice(F0,strike,r,T,ComputedVols);
BlackScholesPrice2=blsprice(F0,strike2,r,T2,ComputedVols2);
BlackScholesPrice3=blsprice(F0,strike3,r,T3,ComputedVols3);
BlackScholesPrice4=blsprice(F0,strike4,r,T4,ComputedVols4);

bspriceSABR = BlackScholesPrice.';
bspriceSABR2 = BlackScholesPrice2.';
bspriceSABR3 = BlackScholesPrice3.';
bspriceSABR4 = BlackScholesPrice4.';

NewDifference= [bspriceSABR'-priceoption, bspriceSABR2'-priceoption2, bspriceSABR3'-priceoption3,
bspriceSABR4'-priceoption4];

figure; surf(t,w,NewDifference)
xlabel('Months to maturity', 'FontWeight', 'bold')
ylabel('Moneyness', 'FontWeight', 'bold')
zlabel('SABR Price - Market Price', 'FontWeight', 'bold')
colorbar
```

```
%Market Implied volatilities

a = xlsread( 'C.xlsx', 'IV Call -5 to -1');

strike=a(:,1);
impvol=a(:,3);
impvol2=a(:,5);
impvol3=a(:,7);
impvol4=a(:,9);

t = [1.5, 5.5, 9.5, 21.5];
z = [impvol, impvol2, impvol3, impvol4];

figure; surf(t,w,z)

xlabel('Months to maturity', 'FontWeight', 'bold')
ylabel('Moneyness', 'FontWeight', 'bold')
zlabel('Market Implied Volatility', 'FontWeight', 'bold')
colorbar

%Monte Carlo Implied volatilities and Prices

x = xlsread( 'C.xlsx', 'MC Call -5 to -1');

impvol=x(:,2);
impvol2=x(:,3);
impvol3=x(:,4);
impvol4=x(:,5);

MCprice=x(:,8);
MCprice2=x(:,9);
MCprice3=x(:,10);
MCprice4=x(:,11);

v = [impvol, impvol2, impvol3, impvol4];
NewDifference2= [MCprice-priceoption, MCprice2-priceoption2, MCprice3-priceoption3, MCprice4-priceoption4];

figure; surf(t,w,v)

xlabel('Months to maturity', 'FontWeight', 'bold')
ylabel('Moneyness', 'FontWeight', 'bold')
zlabel('Monte Carlo Implied Volatility', 'FontWeight', 'bold')
colorbar

figure; surf(t,w,NewDifference2)

xlabel('Months to maturity', 'FontWeight', 'bold')
ylabel('Moneyness', 'FontWeight', 'bold')
zlabel('Monte Carlo Price - Market Price', 'FontWeight', 'bold')
colorbar
```

The following figures portray the respective Excel worksheets to make the previous work, with all the previously stated parameters:

Figure 305: Heston and SABR Call -5 to -1

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
1	35	2.00	5.575	44.34%	61.00	35	6.00	7.9	44.80%	180.00	35	10.00	7.625	36.17%	300.00	35	22.00	9.625	36.34%	665.00
2	40	2.00	1.865	31.79%	61.00	40	6.00	3.3	30.07%	180.00	40	10.00	4.45	30.32%	300.00	40	22.00	6.55	33.61%	665.00
3	45	2.00	0.275	27.15%	61.00	45	6.00	1.24	26.42%	180.00	45	10.00	2.22	27.33%	300.00	45	22.00	4.25	29.60%	665.00
4	50	2.00	0.03	27.49%	61.00	50	6.00	0.33	23.95%	180.00	50	10.00	0.93	25.08%	300.00	50	22.00	2.635	26.86%	665.00
5	55	2.00	0.01	33.60%	61.00	55	6.00	0.09	22.71%	180.00	55	10.00	0.32	23.20%	300.00	55	22.00	1.425	25.75%	665.00

Figure 306: MC Call -5 to -1

	A	B	C	D	E	F	G	H	I	J	K
1	35	0.68952528	0.49854527	0.26630968	0.33524206			5.41647028	6.00351894	6.04223266	7.50199467
2	40	0.70087195	0.502048	0.26558573	0.33621398			1.79791935	2.88593265	2.94053586	4.84939301
3	45	0.697938	0.50156491	0.26419019	0.33629244			0.3165735	1.1421134	1.18543536	3.0040197
4	50	0.68508237	0.49940221	0.26276021	0.33577453			0.02673675	0.37663012	0.40224103	1.79506299
5	55	0.66824035	0.49410989	0.26107942	0.33486195			0.00103035	0.10272002	0.11689272	1.04055707

Figure 307: IV Call -5 to -1

	A	B	C	D	E	F	G	H	I
1	35	5.575	36.77%	7.9	48.80%	7.625	34.55%	9.625	34.69%
2	40	1.865	31.15%	3.3	30.08%	4.45	30.71%	6.55	31.33%
3	45	0.275	28.61%	1.24	27.10%	2.22	27.91%	4.25	29.20%
4	50	0.03	31.06%	0.33	25.10%	0.93	25.78%	2.635	27.28%
5	55	0.01	38.28%	0.09	25.68%	0.32	24.32%	1.425	25.64%

The following two codes are “hidden” codes on which the code previously given in this appendix is based on:

Code 1 - "CF_SVj":

```
function fj=CF_SVj(xt,vt,tau,mu,a,uj,bj,rho,sig,phi)

xj = bj - rho.*sig.*phi.*i;
dj = sqrt( xj.*xj - (sig.*sig).*( 2.*uj.*phi.*i - phi.*phi ) );
gj = ( xj+dj )./( xj-dj );
D = ( xj+dj )./(sig.*sig).* ( 1-exp(dj.*tau) )./( 1-gj.*exp(dj.*tau) ) ;
xx = ( 1-gj.*exp(dj.*tau) )./( 1-gj );
C = mu.*phi.*i.*tau + a./( sig.*sig ) .* ( (xj+dj) .* tau - 2.*log(xx) );
fj = exp( C + D.*vt + i.*phi.*xt );
```

Code 2 - "costf2":

```
function [cost]=costf2(x)
global impvol; global strike; global T; global F0; global r;

for i=1:length(T)
cost(i)=blsprice(F0,strike(i),r,T(i),impvol(i))-HestonCall(F0,strike(i),r,T(i),x(1),x(2),x(3),x(4),x(5),0);
end
```

Code 3 - "HestonCall":

```
function C=HestonCall(St,K,r,T,vt,kap,th,sig,rho,lda)

dphi=0.01;
maxphi=50;
phi=(eps:dphi:maxphi)';

f1 = CF_SVj(log(St),vt,T,0,kap*th,0.5,kap+lda-rho*sig,rho,sig,phi);
P1 = 0.5+(1/pi)*sum(real(exp(-i*phi*log(K)).*f1./(i*phi))*dphi);
f2 = CF_SVj(log(St),vt,T,0,kap*th,-0.5,kap+lda,rho,sig,phi);
P2 = 0.5+(1/pi)*sum(real(exp(-i*phi*log(K)).*f2./(i*phi))*dphi);
C = St*P1 -K*exp(-r*T)*P2;
```